Longtime behavior of the coupled traveling wave model for semiconductor lasers

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Abstract

The coupled traveling wave model is a popular tool for investigating longitudinal dynamical effects in semiconductor lasers, for example, sensitivity to delayed optical feedback. This model consists of a hyperbolic linear system of partial differential equations with one spatial dimension, which is nonlinearly coupled with a slow subsystem of ordinary differential equations. We first prove the basic statements about the existence of solutions of the initial-boundary-value problem and their smooth dependence on initial values and parameters. Hence, the model constitutes a smooth infinite-dimensional dynamical system. Then we exploit this fact and the particular slow-fast structure of the system to construct a low-dimensional attracting invariant manifold for certain parameter constellations. The flow on this invariant manifold is described by a system of ordinary differential equations that is accessible to classical bifurcation theory and numerical tools like such as AUTO.

Key words: laser dynamics, invariant manifold theory, strongly continuous semigroup

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1 Introduction

Semiconductor lasers are known to be extremely sensitive to delayed optical feedback. Even small amounts of feedback may destabilize the laser and cause a variety of nonlinear effects. Self-pulsations, excitability, coexistence

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of several stable regimes, and chaotic behavior have been observed both in experiments and in numerical simulations [1], [2], [3], [4], [5], [6]. Due to their inherent speed, semiconductor lasers are of great interest for modern optical data transmission and telecommunication technology if these nonlinear feedback effects can be cultivated and controlled. Potential applications include, for example, clock recovery [7], [8], generation of pulse trains [9] or high-frequency oscillations [10], and pulse reshaping [11].

Typically, these applications utilize the laser in a non-stationary mode, for example, to produce high-frequency oscillations or pulse trains. Multi-section DFB (distributed feedback) lasers allow one to engineer these nonlinear effects by designing the longitudinal structure of the device [4], [12]. If mathematical modeling is to be helpful in guiding this difficult and expensive design process it has to use models that are, on one hand, as accurate as possible and, on the other hand, give insight into the nature of the observed nonlinear phenomena. The latter is only possible by a detailed bifurcation analysis, while only models involving partial differential equations (PDEs) describe the effects with the necessary accuracy.

We focus in this paper on the *coupled traveling wave model* with gain dispersion. This model is a system of PDEs (one-dimensional in space) coupled to ordinary differential equations (ODEs). It is accurate enough to show quantitatively good correspondence with the experiments and more detailed models [13,14,6]. We prove in this paper that the model can be reduced to a low-dimensional system of ODEs analytically. This makes the model accessible to well-established and powerful numerical bifurcation analysis tools such as AUTO [15]. This in turn allows us to construct detailed and accurate numerical bifurcation diagrams for many practically relevant situations; see [16], [6] for recent results and section 7 for an example.

We achieve the central goal of our paper, the proof of the model reduction, in several steps. First, we show that the PDE system establishing the traveling wave model is a smooth infinite-dimensional dynamical system, that is, it generates a semiflow that is strongly continuous in time and smooth with respect to initial values and parameters. Then, we exploit the particular structure of the model which is of the form

$$\begin{aligned} \dot{E} &= H(n)E \\ \dot{n} &= \varepsilon f(n, |E|) \end{aligned} \tag{1}$$

where the light amplitude $E \in \mathbb{L}^2([0, L]; \mathbb{C}^4)$ is infinite-dimensional and the effective carrier density $n \in \mathbb{R}^m$ is finite-dimensional. The small parameter ε expresses that the carrier density n operates on a much slower time-scale than E. Hence, we investigate in the second step the spectral properties of the linear differential operator H for fixed n and how the growth properties of the semigroup generated by H depend on the spectrum of H. In the last step we construct a low-dimensional invariant manifold for small ε using the general theory on the persistence and properties of normally hyperbolic invariant manifolds for strongly continuous semiflows in Banach spaces [17], [18], [19].

The paper is organized as follows. In Section 2, we introduce the coupled traveling wave model as described in [20] and explain the physical background of all variables and parameters. Section 3 summarizes the results of the paper in a non-technical but precise fashion. It points out the difficulties and the methods and theory used in the proofs. In Section 4 we formulate the PDE system as an abstract evolution equation in a Hilbert space and prove that it establishes a smooth infinite-dimensional system in this setting. In this section, we consider also inhomogeneous boundary conditions in (1) modeling optical injection into the laser. In Section 5 we investigate the spectral properties of the operator H for fixed n and periodic or Dirichlet type boundary conditions, thus, extending results of [21] and [22]. Section 6 is concerned with the construction of a finite-dimensional attracting invariant manifold, where we make use of the slow-fast structure of (1) and the results of Section 4 and Section 5.

Finally, in Section 7 we explain how the system of ODEs obtained in Section 6 can be made accessible to standard numerical bifurcation analysis tools like AUTO. We present a numerical bifurcation diagram for a particluar configuration as an example to demonstrate the usefulness of the model reduction. Moreover, we extend the model reduction theorem of Section 6 to the Lang-Kobayashi system, a delay-differential equation, which is a popular model for a single-mode laser subject to delayed optical feedback from one external reflection [23].

2 The coupled traveling wave model with nonlinear gain dispersion

The *coupled traveling wave model*, a hyperbolic system of PDEs coupled with a system of ODEs is a well known model describing the longitudinal effects in narrow edge-emitting laser diodes [24], [25], [26]. It has been derived from Maxwell's equations for an electro-magnetic field in a periodically modulated waveguide [24], [20] assuming that transversal and longitudinal effects can be separated. In this section we introduce the corresponding system of differential equations, explain the physical interpretation of its coefficients and specify some physically sensible assumptions about these coefficients.

The dynamics in a multi-section laser is described by the evolution of the following quantities. The variable $\psi(t, z) \in \mathbb{C}^2$ describes the complex amplitude of the slowly varying envelope of the optical field split into a forward and a backward traveling wave. The variable $p(t, z) \in \mathbb{C}^2$ describes the correspond-

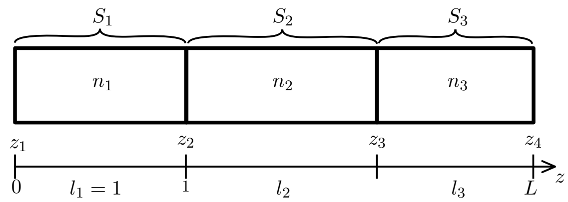

Fig. 1. Typical geometric configuration of the domain in a laser with 3 sections.

ing nonlinear polarization of the material. Both quantities depend on time and the one-dimensional spatial variable $z \in [0, L]$ (the longitudinal direction within the laser; see Figure 1). A prominent feature of multi-section lasers is the splitting of the overall interval $[0, L]$ into sections, that is, m subintervals S_k that represent sections with separate electric contacts. We treat the carrier density within the active zone of the waveguide as a section-wise spatially averaged quantity $n(t) \in \mathbb{R}^m$ (see Fig. 1). In dimensionless form the initial-boundary value problem for ψ, p, and n reads as:

$$\partial_t \psi(t,z) = \begin{bmatrix} -\partial_z + \beta(n(t),z) & -i\kappa(z) \\ -i\kappa(z) & \partial_z + \beta(n(t),z) \end{bmatrix} \psi(t,z) + \rho(n(t),z)\, p(t,z) \quad (2)$$

$$\partial_t p(t,z) = [i\Omega_r(n(t),z) - \Gamma(n(t),z)] \cdot p(t,z) + \Gamma(n(t),z)\, \psi(t,z) \quad (3)$$

$$\begin{aligned} \frac{d}{dt} n_k(t) = I_k - \frac{n_k(t)}{\tau_k} - \frac{P}{l_k} \left[G_k(n_k(t)) - \rho_k(n_k(t))\right] \int_{S_k} \psi(t,z)^* \psi(t,z)\, dz \\ - \frac{P}{l_k} \rho_k(n_k(t)) \,\mathrm{Re}\left(\int_{S_k} \psi(t,z)^* p(t,z)\, dz \right) \text{ for } k = 1 \dots m \end{aligned} \quad (4)$$

subject to the inhomogeneous boundary conditions for ψ

$$\psi_1(t,0) = r_0 \psi_2(t,0) + \alpha(t),\ \psi_2(t,L) = r_L \psi_1(t,L) \quad (5)$$

and the initial conditions

$$\psi(0,z) = \psi^0(z),\ p(0,z) = p^0(z),\ n(0) = n^0. \quad (6)$$

The Hermitian transpose of a $\mathbb{C}^2$-vector ψ is denoted by ψ^* in (4). We will define the appropriate function spaces and discuss the possible solution concepts in sections 3 and 4. The quantities and coefficients appearing above have the following meaning (see also Tab. 1 and Fig. 1). L is the length of the laser. The laser is subdivided into m sections S_k of length l_k with starting points z_k for $k = 1 \dots m$. We scale the system such that $l_1 = 1$ and denote $z_{m+1} = L$. Thus, $S_k = [z_k, z_{k+1}]$. All coefficients are supposed to be spatially constant in each section and to depend only on the carrier density in that section, that is,

	typical range	explanation
$\psi(t,z)$	$\mathbb{C}^2$	optical field, forward and backward traveling wave
$i \cdot p(t,z)$	$\mathbb{C}^2$	nonlinear polarization
$n_k(t)$	$(\underline{n}, \infty)$	spatially averaged carrier density in section S_k in multiples of the transparency carrier density
$\operatorname{Im} \beta_k^0$	$\mathbb{R}$	frequency detuning
$\operatorname{Re} \beta_k^0$	< 0, $O(1)$	decay rate due to internal losses
$\alpha_{H,k}$	$(0, 10)$	negative of line-width enhancement factor
$\tilde{g}_k$	≈ 1	differential gain in active sections S_k
κ_k	$(-10, 10)$	real coupling coefficients for the optical field ψ due to Bragg grating in DFB sections
ρ_k	≥ 0, $O(1)$	maximum of the gain curve
Γ_k	$O(10^2)$	half width of half maximum of the gain curve
$\Omega_{r,k}$	$O(10)$	resonance frequency
I_k	$O(10^{-2})$	current injection
τ_k	$O(10^2)$	spontaneous lifetime for the carriers
P	$(0, \infty)$	scale of (ψ, p) (can be chosen arbitrarily)
r_0, r_L	$\mathbb{C}$, $\lvert r_0 \rvert, \lvert r_L \rvert < 1$	facet reflectivities

Table 1
Ranges and explanations of the variables and coefficients appearing in (2)-(18). See also [20], [27] to inspect their relations to the originally used physical quantities and scales.

if $z \in S_k$,

$$\begin{aligned} \kappa(z) = &\ \kappa_k, & \Gamma(n,z) = &\ \Gamma_k(n_k), \\ \beta(n,z) = &\ \beta_k(n_k), & \rho(n,z) = &\ \rho_k(n_k). \end{aligned}$$

Table 1 collects the physical interpretation and the sensible ranges of all coefficients and variables. The model for the growth coefficient $\beta_k(n_k) \in \mathbb{C}$ in section S_k is

$$\beta_k(\nu) = d_k + (1 + i\alpha_{H,k})G_k(\nu) - \rho_k(\nu)$$

where $d_k \in \mathbb{C}$ accounts for the internal losses (hence, $\operatorname{Re} d_k < 0$) and the frequency detuning, and $\alpha_{H,k} \in \mathbb{R}$ is the negative of the linewidth enhancement (or Henry) factor. A section S_k is either *passive*, then the functions G_k and ρ_k are identically zero, or S_k is *active*. In the active case $G_k : (\underline{n}, \infty) \to \mathbb{R}$

is a smooth strictly monotone increasing function satisfying $G_k(1) = 0$ and $G_k'(1) > 0$. Its limits are $\lim_{\nu\searrow\underline{n}} G_k(\nu) = -\infty$, $\lim_{\nu\to\infty} G_k(\nu) = \infty$. We assume that $\underline{n} \leq 0$ for the lower limit point $\underline{n}$ of G_k. Typical models for G_k in active sections are

$$\begin{aligned} G_k(\nu) = &\ \tilde{g}_k \log \nu && (\underline{n} = \ 0) \text{ or} \\ G_k(\nu) = &\ \tilde{g}_k \cdot (\nu - 1) && (\underline{n} = \ -\infty) \end{aligned}$$

with a differential gain $\tilde{g}_k = G_k'(1) > 0$. In active sections S_k, that is, if $G_k \not\equiv 0$, the gain maximum $\rho_k(\nu)$ is bounded for $\nu < 1$. Moreover, we suppose that ρ_k, $\Omega_{r,k}$, and $\Gamma_k : (\underline{n}, \infty) \to \mathbb{R}$ are smooth and Lipschitz continuous, and $\Gamma_k(\nu) > 1$. For passive sections S_k the variable n_k is decoupled from all other equations and can be dropped from the system.

A remark about the meaning of the quantities p, ρ, Ω_r and Γ: System (2)–(3) models the gain curve of the waveguide material as a Lorentzian. That is, a monochromatic light-wave $\psi_1(t, z) = e^{i\omega t}\varphi(z)$ in an uncoupled and stationary waveguide ($\kappa = 0$, $\dot{n} = 0$) is amplified according to the equation

$$\partial_z |\varphi(z)|^2 = [2 \operatorname{Re} \beta(z) + 2 \operatorname{Re} \chi(i\omega, z)] \ |\varphi(z)|^2$$

where

$$\chi(i\omega, z) = \frac{\rho(z)\Gamma(z)}{i\omega - i\Omega_r(z) + \Gamma(z)}.$$

Hence, ρ is the maximum, Ω_r the location of the maximum, and Γ the half width at half maximum of the gain curve $\operatorname{Re} \chi(i\omega)$ of the waveguide material. The polarization has been included into the coupled traveling wave model for a more realistic account of nonlinear gain dispersion effects [20], [27].

The facet reflectivities r_0 and r_L in (5) are complex with modulus less than 1. The inhomogeneity $\alpha(t)$ is complex and models optical input at the facet $z = 0$. We assume it to be $\mathbb{L}^2$ in time on finite time intervals to permit discontinuous optical input.

The form of the right-hand-side of the equation (4) for the carrier density can be clarified by introducing the Hermitian form

$$g_k(\nu)\left[\begin{pmatrix}\psi\\ p\end{pmatrix}, \begin{pmatrix}\varphi\\ q\end{pmatrix}\right] = \frac{1}{l_k}\int_{S_k} (\psi^*(z), p^*(z)) \begin{pmatrix} G_k(\nu) - \rho_k(\nu) & \frac{1}{2}\rho_k(\nu) \\ \frac{1}{2}\rho_k(\nu) & 0 \end{pmatrix} \begin{pmatrix}\varphi(z)\\ q(z)\end{pmatrix} dz. \quad (7)$$

Using the notation

$$f_k(\nu, (\psi, p)) = I_k - \frac{\nu}{\tau_k} - P g_k(\nu) \left[\begin{pmatrix} \psi \\ p \end{pmatrix}, \begin{pmatrix} \psi \\ p \end{pmatrix} \right] \tag{8}$$

for $\nu \in (\underline{n}, \infty)$ and ψ,φ, p, $q \in \mathbb{L}^2(S_k; \mathbb{C}^2)$ the carrier density equation (4) reads

$$\frac{d}{dt} n_k = f_k(n_k, (\psi, p)) \quad \text{for } k = 1 \ldots m. \tag{9}$$

3 Non-technical overview

In this section we state the main results of the paper in a non-technical but precise manner and summarize the methods used in the proofs of these results. We have split this section into four parts. First we show that system (2)–(4) generates a smooth infinite-dimensional dynamical system. Then we introduce a small parameter. In the next step we investigate the dynamics of the (linear) infinite-dimensional fast subsystem, and finally we construct a low-dimensional attracting invariant manifold.

3.1 Existence theory

In a first step we investigate in which sense system (2)–(4) generates a semiflow depending smoothly on its initial values and all parameters; for details see section 4. We want to write (2)–(4) as an abstract evolution equation in the form

$$\frac{d}{dt} u = Au + g(u)$$

in a Hilbert space V where A is a linear differential operator that generates a strongly continuous semigroup $S(t)$ and g is smooth in V. A natural space for the variables ψ and p is $\mathbb{L}^2([0, L]; \mathbb{C}^2)$, such that V could be $\mathbb{L}^2([0, L]; \mathbb{C}^2) \times \mathbb{L}^2([0, L]; \mathbb{C}^2) \times \mathbb{R}^m$ for the variable $u = (\psi, p, n)$. However, the inhomogeneity α in the boundary condition (5) poses a conceptual difficulty in this framework. Common workarounds are boundary homogenization (used in [8]) or appending α as an auxiliary variable and an additional equation of the form

$$\frac{d}{dt} \alpha(t) = a(t)$$

where a is the derivative of α (used in [28]). Then, the nonlinearity g in the evolution equation depends explicitly on t and it has the same regularity with respect to t as the time derivative of α. Hence, both approaches require a high degree of regularity of α in time which is quite unnatural as the laser still works with discontinuous input. An alternative would be the introduction of a concept of "weakly mild" solutions as was done in [29]. However, this would require the extension of all needed classical results of the theory of strongly continuous semigroups to this type of solutions.

Here, we choose an approach that is similar to that in [28] but does not require any regularity of the inhomogeneity. We introduce the auxiliary space-dependent variable $a(t,x)$ ($x \in [0,\infty)$) satisfying the equation

$$\partial_t a(t,x) = \partial_x a(t,x) \tag{10}$$

and change the boundary condition for $z = 0$ in (5) into

$$\psi_1(t,0) = r_0\psi_2(t,0) + a(t,0).$$

One may think of an infinitely long fibre $[0,\infty)$ storing all future optical inputs and transporting them to the laser facet $z = x = 0$ by the transport equation (10). If we choose $a(0,x) = \alpha(x)$ as initial value for a the value of a at the boundary $x = 0$ at time t is $\alpha(t)$. In this way, the formerly inhomogeneous boundary condition becomes linear in the variables ψ and a requiring no regularity for a. To keep the space V a Hilbert space, we choose a weighted $\mathbb{L}^2$ norm for a that contains $\mathbb{L}^\infty$, that is, $\|a(t,\cdot)\|^2 = \int_0^\infty |a(t,x)|^2(1+x^2)^\eta\, dx$ with $\eta < -1/2$.

With this modification we can work within the framework of the theory of strongly continuous semigroups [30]. The variable u has the components $(\psi, p, n, a) \in V = \mathbb{L}^2([0,L];\mathbb{C}^2) \times \mathbb{L}^2([0,L];\mathbb{C}^2) \times \mathbb{R}^m \times \mathbb{L}^2_\eta([0,\infty);\mathbb{C})$. We have a certain freedom how to choose the splitting of the right-hand-side between A and g. We keep A as simple as possible, including only the unbounded terms

$$A \begin{bmatrix} \psi \\ p \\ n \\ a \end{bmatrix} := \begin{bmatrix} \begin{bmatrix} -\partial_z\psi_1 \\ \partial_z\psi_2 \end{bmatrix} \\ 0 \\ 0 \\ \partial_x a \end{bmatrix}.$$

In this way, it is easy to prove that A generates a strongly continuous semigroup $S(t)$ by constructing S explicitly. The nonlinearity g is smooth because it is a superposition operator of smooth coefficient functions, and all components either depend only linearly on the infinite-dimensional components ψ and p, or map into $\mathbb{R}^m$. Then, the existence of a semiflow $S(t;u)$ that is strongly continuous in t and smooth with respect to u and parameters follows from an a-priori estimate. This a-priori estimate has to be slightly more subtle than in [8]. It uses the fact that the same functions G_k and ρ_k appear on the right-hand-side of (2) and on that of (4) but with opposing signs. Due to this fact we can show that the function

$$\frac{P}{2}\|\psi(t)\|^2 + \sum_{k=1}^{m} l_k(n_k(t) - n_*)$$

remains non-negative for sufficiently small n_* and, hence, bounded, giving rise to a bounded invariant ball in V.

3.2 Introduction of a small parameter

For all results about the long-time behavior of system (2)–(4) we restrict ourselves to autonomous boundary conditions for ψ, that is,

$$\psi_1(t,0) = r_0\psi_2(t,0), \quad \psi_2(t,L) = r_L\psi_1(t,L). \tag{11}$$

The inhomogeneous case is an open question for future work. However, understanding the dynamics of the autonomous laser is not only an intermediate step but an important goal in itself since many experiments and simulations focus on this case; see for example [13] for further references.

Examination of system (2)–(4) reveals that the space dependent subsystem is linear in ψ and p:

$$\partial_t \begin{pmatrix} \psi \\ p \end{pmatrix} = H(n) \begin{pmatrix} \psi \\ p \end{pmatrix}. \tag{12}$$

The linear operator

$$H(n) = \begin{pmatrix} \begin{bmatrix} -\partial_z + \beta(n) & -i\kappa \\ -i\kappa & \partial_z + \beta(n) \end{bmatrix} & \rho(n) \\ \Gamma(n) & i\Omega_r(n) - \Gamma(n) \end{pmatrix} \tag{13}$$

acts from

$$Y := \{(\psi, p) \in \mathbb{H}^1([0,L];\mathbb{C}^2) \times \mathbb{L}^2([0,L];\mathbb{C}^2) : \psi \text{ satisfying (11)}\}$$

into $X = \mathbb{L}^2([0,L];\mathbb{C}^2) \times \mathbb{L}^2([0,L];\mathbb{C}^2)$. $H(n)$ generates a C_0 semigroup $T_n(t)$ acting in X. Its coefficients κ, and, for each $n \in \mathbb{R}^m$, $\beta(n)$, $\Omega_r(n)$, $\Gamma(n)$ and $\rho(n)$ are linear operators in $\mathbb{L}^2([0,L];\mathbb{C}^2)$ defined by the corresponding coefficients in (2), (3). The maps $\beta, \rho, \Gamma, \Omega_r : \mathbb{R}^m \to \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2))$ are smooth.

Furthermore, we observe that I_k and τ_k^{-1} in (8) are approximately two orders of magnitude smaller than 1 (see Tab. 1). Hence, we can introduce a small parameter ε and set $P = \varepsilon$ in (4), such that the carrier density equation (9) reads as

$$\frac{d}{dt}n_k = f_k\left(n_k, E\right) = \varepsilon(F_k(n_k) - g_k(n_k)[E,E]) \tag{14}$$

for $E \in X$ where the coefficients in $F_k(n_k) = \varepsilon^{-1}(I_k - n_k\tau_k^{-1})$ are of order 1. Although ε is not directly accessible, we treat it as a parameter and consider the limit $\varepsilon \to 0$ while keeping F_k fixed. At $\varepsilon = 0$, the carrier density n is constant. It enters the linear subsystem (12) as a parameter. Consequently, the spectral properties of $H(n)$ with fixed n determine the longtime behavior of the system for $\varepsilon = 0$. In particular, we are interested in n where an isolated non-empty but finite set of eigenvalues of $H(n)$ is located exactly on the imaginary axis. In this case, we can expect a finite-dimensional invariant manifold to persist for nonzero ε in the spirit of Fenichel's geometric singular perturbation theory [33]. Thus, we would like to understand the spectral properties of the operator H for fixed n and their correspondence to the growth of the semigroup T generated by H in the next step.

3.3 Spectral properties of $H(n)$

We drop the argument n in this paragraph for brevity. The goal of this part is to show that (for realistic n) we can find a rate $\xi < 0$ and a splitting of $X = X_1 \oplus X_2$ into two H-invariant subspaces where X_1 is finite-dimensional and the semigroup T restricted to X_2 decays with rate ξ:

$$\|T(t)\| \leq Me^{\xi t} \quad \text{for a constant } M \geq 1 \text{ and all } t \geq 0;$$

for details see section 5. Since T is not an analytical or eventually compact semigroup there are no general theorems implying our result. However, the operator H has a characteristic function $h(\lambda)$ defined in $\mathbb{C} \setminus \mathcal{W}$ where $\mathcal{W} = \{i\Omega_{r,k} - \Gamma_k : k = 1, \ldots, m\}$ (note that $\operatorname{Re}\mathcal{W} < -1$). The function h is analytic in $\mathbb{C} \setminus \mathcal{W}$ and known explicitly. Hence, most questions about the spectrum of H can be answered by finding the roots of h. In particular, the spectrum of H is discrete in $\mathbb{C} \setminus \mathcal{W}$, that is, it consists only of eigenvalues of finite algebraic multiplicity. In order to obtain our result, we have to distinguish two cases, $r_0 r_L = 0$ (that is, (11) are Dirichlet boundary conditions) and $r_0 r_L \neq 0$ (periodic boundary conditions).

It turns out that the semigroup T is eventually differentiable if $r_0 r_L = 0$. In this case, we can split X into two H-invariant subspaces. One corresponds to the spectrum close to $\mathcal{W}$. Thus, H is bounded and T decaying in this subspace. The semigroup T restricted to the complementary invariant subspace is eventually compact. Hence, the desired result follows from the theory of eventually compact semigroups [31].

If $r_0 r_L \neq 0$ (the hyperbolic case), we treat the operator as a perturbation of its diagonal part similar to [21]. Before applying the same result as [21], the invariant subspace corresponding to the spectrum close to $\mathcal{W}$ has to be split off and treated separately in the same way as in the case $r_0 r_L = 0$.

In essence, the result of section 5 implies that we can treat H like a matrix: The dominant eigenvalues determine the growth of the corresponding semigroup.

3.4 Existence of a low-dimensional invariant manifold

Let us assume that there exists a simple connected open set $U \subset \mathbb{R}^m$ of carrier densities n such that $H(n)$ has a uniform spectral gap for all $n \in U$ in a strip of the negative complex half-plane $\{z \in \mathbb{C} : \xi \le \operatorname{Re} z \le \xi/k\}$ ($\xi < 0$, integer $k > 2$), and that the dominant part of the spectrum of $H(n)$ is finite. Hence, the spectral projection $P_c(n)$ onto the $H(n)$-invariant subspace corresponding to the dominant part of the spectrum has constant rank q. This spectral gap assumption is quite natural and follows for example from the existence of non-trivial dynamics that is uniformly bounded for $\varepsilon \to 0$ (e.g., relative equilibria, i.e., solutions of the form $E(t) = E_0 e^{i\omega t}$, $n = \text{const}$) if $r_0 r_L = 0$. We can split any $E \in X$ into $E = B(n)E_c + E_s$ where $B(n)$ is a basis of $\operatorname{Im} P_c(n)$ depending smoothly on n, $E_c \in \mathbb{C}^q$, and $E_s \in X$ is $E - P_c(n)B(n)E_c$. The map $\mathcal{R} : X \times U \to \mathbb{C}^q \times U$ given by $(E, n) \to (B(n)^{-1}P_c(n)E, n)$ is well defined, smooth and Lipschitz continuous on any closed subset of $X \times U$. Then, the main model reduction theorem is as follows.

Theorem 1 (Model reduction)
Let $\varepsilon_0 > 0$ be sufficiently small, $\Delta \in (\xi, 0)$, and $\mathcal{N}$ be a closed bounded subset of $\mathbb{C}^q \times U$. Then, for all $\varepsilon \in [0, \varepsilon_0)$ there exists a C^k manifold $\mathcal{C} \subset X \times \mathbb{R}^m$ satisfying:

(i) (Invariance) *$\mathcal{C}$ is $S(t,\cdot)$-invariant relative to $\mathcal{R}^{-1}\mathcal{N}$. That is, if $(E,n) \in \mathcal{C}$, $t \ge 0$, and $S([0,t];(E,n)) \subset \mathcal{R}^{-1}\mathcal{N}$, then $S([0,t];(E,n)) \subset \mathcal{C}$.*

(ii) (Representation) *$\mathcal{C}$ can be represented as the graph of a map which maps*

$$(E_c, n, \varepsilon) \in \mathcal{N} \times [0, \varepsilon_0) \to ([B(n) + \varepsilon\nu(E_c, n, \varepsilon)]E_c, n) \in X \times \mathbb{R}^m$$

where $\nu : \mathcal{N} \times [0, \varepsilon_0) \to \mathcal{L}(\mathbb{C}^q; X)$ is C^{k-2} with respect to all arguments. Denote the X-component of $\mathcal{C}$ by

$$E_X(E_c, n, \varepsilon) = [B(n) + \varepsilon\nu(E_c, n, \varepsilon)]E_c \in X.$$

(iii) (Exponential attraction) *Let $\Upsilon \subset X \times \mathbb{R}^m$ be a bounded set with $\mathcal{R}\Upsilon \subset \mathcal{N}$ and a positive distance to the boundary of $\mathcal{N}$. Then, there exist a constant M and a time $t_c \ge 0$ with the following property: For any $(E,n) \in \Upsilon$ there*

exists a $(E_c, n_c) \in \mathcal{N}$ *such that*

$$\|S(t + t_c; (E, n)) - S(t; (E_X(E_c, n_c, \varepsilon), n_c))\| \leq M e^{\Delta t}$$

for all $t \geq 0$ *with* $S([0, t + t_c]; (E, n)) \subset \Upsilon$*.*

(iv) (Flow) *The flow on* $\mathcal{C} \cap \mathcal{R}^{-1}\mathcal{N}$ *is differentiable with respect to* t *and governed by the following system of ODEs:*

$$\begin{aligned} \frac{d}{dt} E_c &= \Big[H_c(n) + \varepsilon a_1(E_c, n, \varepsilon) + \varepsilon^2 a_2(E_c, n, \varepsilon) \nu(E_c, n, \varepsilon) \Big] E_c \\ \frac{d}{dt} n &= \varepsilon F(E_c, n, \varepsilon) \end{aligned} \tag{15}$$

where

$$\begin{aligned} H_c(n) &= B(n)^{-1} H(n) P_c(n) B(n) \\ a_1(E_c, n, \varepsilon) &= -B(n)^{-1} P_c(n) \partial_n B(n) F(E_c, n, \varepsilon) \\ a_2(E_c, n, \varepsilon) &= B(n)^{-1} \partial_n P_c(n) F(E_c, n, \varepsilon)(Id - P_c(n)) \\ F(E_c, n, \varepsilon) &= (F_k(n_k) - g_k(n_k)[E_X(E_c, n_c, \varepsilon), E_X(E_c, n_c, \varepsilon)])_{k=1}^m . \end{aligned}$$

The idea to choose n-dependent coordinates for E in the construction of a reduced model was introduced already in [24] by physicists. This choice has the advantage that the graph of the center manifold itself enters the flow (15) on the center manifold only in the form $O(\varepsilon^2)\nu$. This fact has been pointed out first in [32] where the same model reduction result has been proven for ODEs of the structure (1) using Fenichel's Theorem for singularly perturbed systems of ODEs [33]. Since Fenichel's Theorem is not available for infinite-dimensional systems, we have to adapt the proof in [33] to our case starting from the general results in [17], [18], [19] about invariant manifolds of semiflows in Banach spaces. In particular, we apply the cut-off modifications done in [33] only to the finite-dimensional components E_c and n outside of the set $\mathcal{N}$ of interest. Moreover, we adapt the modifications such that the invariant manifold for $\varepsilon = 0$ is compact without boundary as required by the theorems in [17].

Truncating all terms of order $O(\varepsilon^2)$ in (15) gives rise to a system of ODEs in $\mathbb{C}^q \times \mathbb{R}^m$ where all terms in the right-hand-side can be expressed analytically as functions of the eigenvalues of H. The truncated system (15) is called the *mode approximation*. It is an implicit system of ODEs because the eigenvalues of H are given only implicitly as roots of the characteristic function h of H. The dimension of (15) is typically low: q is often either 1 or 2. The consideration of mode approximations has proven to be extremely useful for numerical and analytical investigations of longitudinal effects in multi-section semiconductor lasers; see for example [7], [16], [6].

4 Existence and Uniqueness of Classical and Mild Solutions

In this section, we treat the inhomogeneous initial-boundary value problem (2)-(5) as an autonomous nonlinear evolution equation

$$\frac{d}{dt}u(t) = Au(t) + g(u(t)), \quad u(0) = u_0 \tag{16}$$

where $u(t)$ is an element of a Hilbert space V, A is a generator of a C_0 semigroup $S(t)$, and $g : U \subseteq V \to V$ is smooth and locally Lipschitz continuous in an open set $U \subseteq V$. The inhomogeneity in (5) is included in (16) as a component of u.

4.1 Notation

The Hilbert space V is defined by

$$V := \mathbb{L}^2([0, L]; \mathbb{C}^2) \times \mathbb{L}^2([0, L]; \mathbb{C}^2) \times \mathbb{R}^m \times \mathbb{L}^2_\eta([0, \infty); \mathbb{C}) \tag{17}$$

where $\mathbb{L}^2_\eta([0, \infty); \mathbb{C})$ is the space of weighted square integrable functions. The scalar product of $\mathbb{L}^2_\eta([0, \infty); \mathbb{C})$ is defined by

$$(v, w)_\eta := \mathrm{Re} \int_0^\infty \bar{v}(x) \cdot w(x)(1 + x^2)^\eta dx.$$

We choose $\eta < -1/2$ such that the space $\mathbb{L}^\infty([0, \infty); \mathbb{C})$ is continuously embedded in $\mathbb{L}^2_\eta([0, \infty); \mathbb{C})$. The complex plane is treated as two-dimensional real plane in the definition of the vector space V such that the standard $\mathbb{L}^2$ scalar product $(\cdot, \cdot)_V$ of V is differentiable. The corresponding components of $v \in V$ are denoted by

$$v = (\psi, p, n, a).$$

Here, ψ and p have two complex components and $n \in \mathbb{R}^m$. The spatial variable in ψ and p is denoted by $z \in [0, L]$, whereas the spatial variable in a is denoted by $x \in [0, \infty)$. The Hilbert space $\mathbb{H}^1_\eta([0, \infty); \mathbb{C})$ equipped with the scalar product

$$(v, w)_{1,\eta} := (v, w)_\eta + (\partial_x v, \partial_x w)_\eta$$

is densely and continuously embedded in $\mathbb{L}^2_\eta([0, \infty); \mathbb{C})$. Moreover, its elements are continuous [34]. Consequently, the Hilbert spaces

$$\begin{aligned} W &:= \mathbb{H}^1([0, L]; \mathbb{C}^2) \times \mathbb{L}^2([0, L]; \mathbb{C}^2) \times \mathbb{R}^m \times \mathbb{H}^1_\eta([0, \infty); \mathbb{C}), \text{ and} \\ W_{\mathrm{BC}} &:= \{(\psi, p, n, a) \in W : \psi_1(0) = r_0\psi_2(0) + a(0), \psi_2(L) = r_L\psi_1(L)\} \end{aligned}$$

are densely and continuously embedded in V. The linear functionals $\psi_1(0) - r_0\psi_2(0) - a(0)$ and $\psi_2(L) - r_L\psi_1(L)$ are continuous from $W \to \mathbb{R}$. We define the linear operator $A : W_{\mathrm{BC}} \to V$ by

$$A \begin{bmatrix} \psi \\ p \\ n \\ a \end{bmatrix} := \begin{bmatrix} \begin{bmatrix} -\partial_z \psi_1 \\ \partial_z \psi_2 \end{bmatrix} \\ 0 \\ 0 \\ \partial_x a \end{bmatrix}.$$

The definition of A and W_{BC} treat the inhomogeneity α in the boundary condition (5) as the boundary value at 0 of the variable a. We define the open set $U \subseteq V$ by

$$U := \{(\psi, p, n, a) \in V : n_k > \underline{n} \text{ for } k = 1 \ldots m\},$$

and the nonlinear function $g : U \to V$ by

$$g(\psi, p, n, a) = \begin{pmatrix} \beta(n)\psi - i\kappa \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \psi + \rho(n)p \\ (i\Omega_r(n) - \Gamma(n))p + \Gamma(n)\psi \\ (f_k(n_k, (\psi, p)))_{k=1}^m \\ 0 \end{pmatrix}. \tag{18}$$

The corresponding coefficients of (2)–(4) define the smooth maps $\beta : (\underline{n}, \infty)^m \to \mathcal{L}(\mathbb{L}^2([0, L]; \mathbb{C}^2))$ and $\rho, \Omega_r, \Gamma : \mathbb{R}^m \to \mathcal{L}(\mathbb{L}^2([0, L]; \mathbb{C}^2))$. The function g is continuously differentiable to any order with respect to all arguments and its Frechet derivative is bounded in any closed bounded ball $B \subset U$ [28].

According to the theory of C_0 semigroups, there are two solution concepts [30]:

Definition 2 *Let $T > 0$. A solution $u : [0, T] \to V$ is a classical solution of* (16) *if $u(t) \in W_{\mathrm{BC}} \cap U$ for all $t \in [0, T]$, $u \in C^1([0, T]; V)$, $u(0) = u_0$, and equation* (16) *is valid in V for all $t \in (0, T)$.*

The inhomogeneous initial-boundary value problem (2)-(6) and the autonomous evolution system (16) are equivalent in the following sense: Suppose $\alpha \in \mathbb{H}^1([0, T); \mathbb{C})$ in (5). Let $u = (\psi, p, n, a)$ be a classical solution of (16). Then, u satisfies (2)-(3), and (6) in $\mathbb{L}^2$ and (4), (5) for each $t \in [0, T]$ if and only if $a^0|_{[0,T]} = \alpha$. On the other hand, assume that (ψ, p, n) satisfies (2)-(3), and (6) in $\mathbb{L}^2$ and (4), (5) for each $t \in [0,T]$. Then, we can choose a $a^0 \in \mathbb{H}^1_\eta([0,\infty);\mathbb{C})$ such that $a^0|_{[0,T]} = \alpha$ and obtain that $u(t) = (\psi(t), p(t), n(t), a^0(t+\cdot))$ is a classical solution of (16) in $[0,T]$.

Definition 3 *Let $T > 0$, A be a generator of a C_0 semigroup $S(t)$ of bounded operators in V. A solution $u : [0,T] \to V$ is a mild solution of (16) if $u(t) \in U$ for all $t \in [0,T]$, and $u(t)$ satisfies the variation of constants formula in V*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)g(u(s))ds. \tag{19}$$

We prove in Lemma 4 that A generates a C_0 semigroup in V. Mild solutions of (16) are a reasonable generalization of the classical solution concept of (2)-(5) to boundary conditions including discontinuous inputs $\alpha \in \mathbb{L}^2_\eta([0,\infty);\mathbb{C})$.

4.2 Global Existence and Uniqueness of Solutions for the Truncated Problem

In order to prove uniqueness and global existence of solutions of (16), we apply the theory of strongly continuous semigroups [30].

Lemma 4 *$A : W_{\mathrm{BC}} \subset V \to V$ generates a C_0 semigroup $S(t)$ of bounded operators in V.*

PROOF. We specify the C_0 semigroup $S(t)$ explicitly. Denote the components of $S(t)((\psi_1^0, \psi_2^0), p^0, n^0, a^0)$ by $((\psi_1(t,z), \psi_2(t,z)), p(t,z), n(t), a(t,x))$ for $z \in [0,L]$, $x \in [0,\infty)$, and let $t \le L$.

$$\begin{aligned}
\psi_1(t,z) &= \begin{cases} \psi_1^0(z-t) & \text{for} \quad z > t \\ r_0\psi_2^0(t-z) + a^0(t-z) & \text{for} \quad z \le t \end{cases} \\
\psi_2(t,z) &= \begin{cases} \psi_2^0(z+t) & \text{for} \quad z < L-t \\ r_L\psi_1^0(2L-t-z) & \text{for} \quad z \ge L-t \end{cases} \\
p(t,z) &= 0 \\
n(t) &= 0 \\
a(t,x) &= a^0(x+t).
\end{aligned}$$

For $t > L$ we define inductively $S(t)u = S(L)S(t-L)u$. This procedure defines a semigroup of bounded operators in V since

$$\|\psi_1(t,\cdot)\|^2 + \|\psi_2(t,\cdot)\|^2 + \|a(t,\cdot)\|^2 \le 2(1+t^2)^{-\eta}\left(\|\psi_1^0\| + \|\psi_2^0\| + \|a^0\|\right)$$

for $t \leq L$. The strong continuity of S is a direct consequence of the continuity in the mean in $\mathbb{L}^2$. It remains to be shown that S is generated by A.

Let $u = ((\psi_1^0, \psi_2^0), p^0, n^0, a^0)$ satisfy $\lim_{t\to 0} \frac{1}{t}(S(t)u - u) \in V$, define $\varphi_t(z) := \frac{1}{t}(\psi_1(t,z) - \psi_1^0(z))$, $\varphi_0 = \lim_{t\to 0} \varphi_t$, and let $\delta > 0$ be small. Firstly, we prove that $u \in W_{\mathrm{BC}}$. φ_t coincides with the difference quotient $\frac{1}{t}(\psi_1^0(z-t) - \psi_1^0(z))$ for $t < \delta$ and $z \in [\delta, L]$. Thus, $\partial_z \psi_1^0 \in \mathbb{L}^2([\delta, L]; \mathbb{C})$ exists. Furthermore, $\varphi_t(\cdot + t) \to \varphi_0$ in $\mathbb{L}^2([0, L-\delta]; \mathbb{C})$. Since $\varphi_t(\cdot + t) = \frac{1}{t}(\psi_1^0(z) - \psi_1^0(z+t))$, $\partial_z \psi_1^0$ exists also in $\mathbb{L}^2([0, L-\delta]; \mathbb{C})$. Consequently $\psi_1^0 \in \mathbb{H}^1([0,L]; \mathbb{C})$. The same argument holds for $\psi_2^0 \in \mathbb{H}^1([0,L]; \mathbb{C})$ and for $a^0 \in \mathbb{H}^1_\eta([0,\infty); \mathbb{C})$.

In order to verify that u satisfies the boundary conditions we write

$$\varphi_t(z) = \begin{cases} z \in [t, L]: & -\frac{1}{t}\int_{z-t}^{z} \partial_z \psi_1^0(\zeta) d\zeta \\ z \in [0, t]: & \frac{1}{t}\left(r_0 \int_0^{t-z} \partial_z \psi_2^0(\zeta) + \partial_z a^0(\zeta) d\zeta - \int_0^z \partial_z \psi_1^0(\zeta) d\zeta\right) + \\ & +\frac{1}{t}\left(r_0 \psi_2^0(0) + a^0(0) - \psi_1^0(0)\right) \end{cases} \tag{20}$$

Consequently, the limit φ_0 is in $\mathbb{L}^2([0,L]; \mathbb{C})$ if and only if $r_0 \psi_2^0(0) + a^0(0) - \psi_1^0(0) = 0$. The same argument using $\frac{1}{t}(\psi_2(t,z) - \psi_2^0(z))$ implies $r_L \psi_1^0(L) - \psi_2^0(L) = 0$.

Finally, we prove that for any $u \in W_{\mathrm{BC}}$ we have $\lim_{t\to 0} \frac{1}{t}(S(t)u - u) = Au$. Using the notation φ_t introduced above, we have $\int_0^t |\varphi_t(z)|^2 dz \to_{t\to 0} 0$ due to (20). Hence, $\varphi_t \to_{t\to 0} -\partial_z \psi_1^0$ on $[0,L]$. Again, we can use the same arguments to obtain the limits $\partial_z \psi_2^0$ and $\partial_x a^0$. $\square$

The operators $S(t)$ have a uniform upper bound

$$\|S(t)\| \leq C e^{\gamma t} \tag{21}$$

within finite intervals $[0, T]$. In order to apply the results of the C_0 semigroup theory [30], we truncate the nonlinearity g smoothly: For any bounded ball $B \subset U$ which is closed w.r.t. V, we choose $g_B : V \to V$ such that g_B is smooth, globally Lipschitz continuous, and $g_B(u) = g(u)$ for all $u \in B$. This is possible because the Frechet derivative of g is bounded in B and the scalar product in V is differentiable with respect to its arguments. We call

$$\frac{d}{dt}u(t) = Au(t) + g_B(u(t)), \quad u(0) = u_0 \tag{22}$$

the truncated problem (16). The following Lemma 5 is a consequence of the results in [30].

Lemma 5 (global existence for the truncated problem)
The truncated problem (22) *has a unique global mild solution* $u(t)$ *for any* $u_0 \in V$. *If* $u_0 \in W_{\mathrm{BC}}$, $u(t)$ *is a classical solution of* (22).

Corollary 6 (local existence) *Let $u_0 \in U$. There exists a $t_{\mathrm{loc}} > 0$ such that the evolution problem* (16) *has a unique mild solution $u(t)$ on the interval $[0, t_{\mathrm{loc}}]$. If $u_0 \in W_{\mathrm{BC}} \cap U$, $u(t)$ is a classical solution of* (16) *in $[0, t_{\mathrm{loc}}]$.*

4.3 A-priori Estimate — Existence of Semiflow

In order to state the existence result of Lemma 5 for the original system (16), we need the following a-priori estimate for the solutions of the truncated problem (22).

Lemma 7 *Let $T > 0$, $u_0 \in U$. If $\underline{n} > -\infty$, we suppose $I_k \tau_k > \underline{n}$ for all $k = 1 \dots m$. Then, there exists a closed bounded ball B such that $B \subset U$ and the solution $u(t)$ of the B-truncated problem* (22) *starting at u_0 stays in B for all $t \in [0, T]$.*

PROOF. First, let $u_0 = (\psi^0, p^0, n^0, a^0) \in D(A) = W_{\mathrm{BC}} \cap U$.

Preliminary consideration
Let $n_* \in (\underline{n}, n_k^0)$ be such that $G_k(n_*) - \rho_k(n_*) < 0$ for all $k = 1 \dots m$ where $G_k \not\equiv 0$ (i.e., for all active sections S_k). Let $t_1 > 0$ be such that the solution of the non-truncated problem (16) $u(t) = (\psi(t), p(t), n(t), a(t))$ exists in $[0, t_1]$, and $n_k(t) \geq n_*$ for all $k = 1 \dots m$ and $t \in [0, t_1]$.

$$h(t) := \frac{P}{2} \|\psi(t)\|^2 + \sum_{k=1}^{m} l_k (n_k(t) - n_*).$$

Because of the structure of the nonlinearity g, which is linear in ψ in its first component, $u(t)$ is classical in $[0, t_1]$. Hence, $h(t)$ is differentiable and the differential equations (2) and (4) imply

$$\begin{aligned} \frac{d}{dt} h(t) &\leq J + \sup_{z \in \mathbb{C}} \{ |r_0 z + a^0(t)|^2 - |z|^2 \} - \sum_{k=1}^{m} \left(\frac{l_k}{\tau_k} n_k + P \operatorname{Re} d_k \int_{S_k} |\psi(z)|^2 \, dz \right) \\ &\leq J + \frac{|a^0(t)|^2}{1 - |r_0|^2} - \tilde{\tau}^{-1} n_* - \gamma h(t), \end{aligned} \tag{23}$$

where

$$J := \sum_{k=1}^{m} l_k I_k, \quad \tilde{\tau}^{-1} := \sum_{k=1}^{m} l_k \tau_k^{-1}, \quad \gamma := \min \left\{ \tau_k^{-1}, -\frac{\operatorname{Re} d_k}{2} : k = 1 \dots m \right\} > 0.$$

Consequently,

$$
\begin{aligned}
h(t) &\le h(0) + Jt - \tilde{\tau}^{-1} t\, n_* + \frac{1}{1-|r_0|^2}\int_0^t |a^0(s)|^2\, ds \\
&\le h(0) + JT + \tilde{\tau}^{-1} T\, |n_*| + \frac{(1+T^2)^{-\eta}}{1-|r_0|^2}\|a^0\|^2 \\
&\le \left(\frac{P}{2}\|\psi^0\|^2 + \sum_{k=1}^m l_k n_k^0 + JT + \frac{(1+T^2)^{-\eta}}{1-|r_0|^2}\|a^0\|^2\right) + \left(L + \tilde{\tau}^{-1}T\right)\, |n_*| \\
&\le M + \xi\, |n_*| \qquad (24)
\end{aligned}
$$

for all $t \in [0, t_1]$ where M and ξ do not depend on n_*. The inequality (24) remains valid even if $u_0 \in (V \setminus W_{\mathrm{BC}}) \cap U$ (i.e., $u(t)$ is not classical but mild) as both sides of (24) depend only on the V-norm of u but not on its W_{BC}-norm. Since $n_k(t) \ge n_*$ in $[0, t_1]$ for all $k = 1 \dots m$, the estimate (24) for $h(t)$ and the differential equation (3) for p imply bounds for ψ, p and n in $[0, t_1]$:

$$
\begin{aligned}
\|\psi(t)\|^2 &\le S(n_*)^2 := 2P^{-1}(M + \xi\, |n_*|) \\
\|p(t)\| &\le \|p^0\| + S(n_*) \qquad (25) \\
n_k &\in \left[n_*, n_* + (2l_k)^{-1} P S(n_*)^2\right].
\end{aligned}
$$

Hence, $f_k(n_*, (\psi(t), p(t)))$ is greater than

$$
I_k - \frac{n_*}{\tau_k} - \frac{P}{l_k}\max_{\Theta\in\mathbb{R}}\left[(G_k(n_*) - \rho_k(n_*))\Theta^2 + |\rho_k(n_*)|(\|p^0\| + S(n_*))\Theta\right] \qquad (26)
$$

for all $k = 1 \dots m$ and $t \in [0, t_1]$.

Construction of B
Since $G_k(\nu) \to_{\nu\to\underline{n}} -\infty$ and $\rho_k(\nu)$ bounded for $\nu \to \underline{n}$, or $G_k = \rho_k = 0$, we can find a n_* such that the expression (26) is greater than 0 for all $k = 1 \dots m$. Then, we choose B such that $u = (\psi, p, n, a) \in B$ if ψ, p and n satisfy (25) for the chosen n_* and $a = a^0(t + \cdot)$ for $t \in [0, T]$.

Indirect proof of invariance of B
Assume that the solution $v(t) = (\psi(t), p(t), n(t), a(t))$ of the B-truncated problem leaves B. The preliminary consideration and the construction of B imply that there exists a t_1 such that $u(t)$ exists in $[0, t_1]$, and, for one $k \in \{1 \dots m\}$, $n_k(t_1) = n_*$ and $n_k(t) > n_*$ for all $t \in [0, t_1]$. Consequently, $\dot{n}_k(t_1) = f_k(n_k(t_1), (\psi(t_1), p(t_1))) < 0$. However, this contradicts to the construction of n_* such that (26) is greater than 0. □

Lemma 7 implies the following global existence theorem for mild and classical solutions:

Theorem 8 (global existence and uniqueness)
Let $T > 0$, $u_0 \in U$. If $\underline{n} > -\infty$, let $I_k\tau_k > \underline{n}$ for all $k = 1 \dots m$. There exists

a unique mild solution $u(t)$ of (16) *in* $[0,T]$. *Furthermore, if* $u_0 \in W_{\mathrm{BC}} \cap U$, $u(t)$ *is a classical solution of* (16).

If the component a is globally bounded, i.e., $a^0 \in L^\infty$, the ball B constructed in the a-priori estimate of Lemma 7 does not depend on the end T of the time interval either. Thus, the solutions are globally bounded if a^0 is bounded:

Corollary 9 (global boundedness)
Let $u_0 = (\psi^0, p^0, n^0, a^0) \in U$ *and* $\|a^0\|_\infty < \infty$. *There exists a constant* C *such that* $\|u(t)\|_V \leq C$.

PROOF. It is sufficient to prove that the constants M and ξ in the estimate (24) for $h(t)$ do not depend on T if $\|a^0\|_\infty < \infty$. The estimate (23) for $\dot{h}(t)$ implies

$$\begin{aligned}
h(t) &\leq \max\left\{h(0), \frac{1}{\gamma}\left(J + \frac{\|a^0\|_\infty}{1-|r_0|^2} - \frac{n_*}{\tilde{\tau}}\right)\right\} \\
&\leq \left(\frac{P}{2}\|\psi^0\|^2 + \sum_{k=1}^{m} l_k n_k^0 + L|n_*|\right) + \frac{1}{\gamma}\left(J + \frac{\|a^0\|_\infty}{1-|r_0|^2} + \frac{|n_*|}{\tilde{\tau}}\right) \\
&\leq \left(\frac{P}{2}\|\psi^0\|^2 + \sum_{k=1}^{m} l_k n_k^0 + \frac{1}{\gamma}\left[J + \frac{\|a^0\|_\infty}{1-|r_0|^2}\right]\right) + \left(L + \frac{1}{\gamma\tilde{\tau}}\right)|n_*| \\
&\leq M + \xi\,|n_*| \qquad (27)
\end{aligned}$$

where now M and ξ do not depend on T. Hence, the bounds (25) can now be derived from (27) in the same way as in the proof of Lemma 7 using the T-independent bounds M and ξ. Consequently, we can choose n_* independent of T and, hence, the ball B does not depend on T (see proof of Lemma 7). □

Let us define the semiflow map $S : [0,\infty) \times U \to U$ by $S(t;u_0) := u(t)$ where $u(t)$ is the mild solution of the evolution equation (16) with initial value $u(0) = u_0$. The following corollary is an immediate consequence of the general theory of C_0 semigroups [30] and the smoothness of the nonlinearity g in the evolution equation (16):

Corollary 10 (smooth dependence on initial values)
The map $(t, u_0) \to S(t;u_0)$ *is smooth with respect to* u_0 *and strongly continuous with respect to* t.

The smooth dependence of the solution on all parameters within a bounded parameter region is also a direct consequence of the C_0 semigroup theory. The restrictions imposed on the parameters in Section 2 and Lemma 7 have to be satisfied uniformly in the considered parameter range in order to obtain a uniform a-priori estimate. In particular, we point out that the ball in the a-priori estimate of Lemma 7 can be chosen uniform for $I_k \to 0$ (if $\underline{n} < 0$) and $\tau_k^{-1} \to 0$.

5 Asymptotic behavior of the linear part — spectral properties of $H(n)$ for fixed n

We restrict ourselves to the autonomous system (2)–(4) in the following. The boundary conditions are

$$\psi_1(t,0) = r_0\psi_2(t,0), \quad \psi_2(t,L) = r_L\psi_1(t,L) \tag{28}$$

in the autonomous case.

As mentioned in Section 3, the long-time behavior of the overall system at $\varepsilon = 0$ in (14) (i.e., $\dot{n}_k = 0$ for $k = 1\ldots m$) is determined by the behavior of the linear space-dependent subsystem (12), that is, the spectral properties of the operator $H(n)$. In this section we treat n as a parameter, dropping the corresponding argument from the coefficients β, ρ, Ω_r, and Γ for brevity.

Define the set of complex "resonance frequencies"

$$\mathcal{W} = \{c \in \mathbb{C} : c = i\Omega_{r,k} - \Gamma_k \text{ for at least one } k \in \{1\ldots m\}\} \subset \mathbb{C}$$

and $\chi : \mathbb{C}\setminus\mathcal{W} \to \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2))$ (see section 2 for explanation and [20], [27] for details) by

$$\chi(\lambda) = \frac{\rho\Gamma}{\lambda - i\Omega_r + \Gamma} \in \mathcal{L}(\mathbb{L}^2([0,L];\mathbb{C}^2)) \text{ for each } \lambda \in \mathbb{C}\setminus\mathcal{W}.$$

For $\lambda \in \mathbb{C}\setminus\mathcal{W}$, the following relation follows from (13): λ is in the resolvent set of H if and only if the boundary value problem

$$\begin{bmatrix} -\partial_z + \beta + \chi(\lambda) - \lambda & -i\kappa \\ -i\kappa & \partial_z + \beta + \chi(\lambda) - \lambda \end{bmatrix} \varphi = 0$$
$$\text{with b. c.} \quad \varphi_1(t,0) = r_0\varphi_2(t,0), \quad \varphi_2(t,L) = r_L\varphi_1(t,L) \tag{29}$$

has only the trivial solution $\varphi = 0$ in $\mathbb{H}^1([0,L];\mathbb{C}^2)$. The transfer matrix corresponding to (29) is

$$T_k(z,\lambda) = \frac{e^{-\gamma_k z}}{2\gamma_k}\begin{pmatrix} \gamma_k + \mu_k + e^{2\gamma_k z}(\gamma_k - \mu_k) & i\kappa_k\left(1 - e^{2\gamma_k z}\right) \\ -i\kappa_k\left(1 - e^{2\gamma_k z}\right) & \gamma_k - \mu_k + e^{2\gamma_k z}(\gamma_k + \mu_k) \end{pmatrix} \tag{30}$$

for $z \in S_k$ where $\mu_k = \lambda - \chi_k(\lambda) - \beta_k$ and $\gamma_k = \sqrt{\mu_k^2 + \kappa_k^2}$ [24], [21]. The right-hand-side of (30) does not depend on the branch of the square root in γ_k since the expression is even with respect to γ_k. Denote the overall transfer matrix of (29) by $T(z_1, z_2; \lambda)$ for $z_1, z_2 \in [0, L]$. The function

$$h(\lambda) = \begin{pmatrix} r_L, & -1 \end{pmatrix} T(L, 0; \lambda) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} = \begin{pmatrix} r_L & -1 \end{pmatrix} \prod_{k=m}^{1} T_k(l_k; \lambda) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} \tag{31}$$

defined in $\mathbb{C} \setminus \mathcal{W}$ is the characteristic function of H: Its roots are the eigenvalues of H and $\mathcal{R} := \{\lambda \in \mathbb{C} \setminus \mathcal{W} : h(\lambda) \neq 0\}$ is the resolvent set. Consequently, all $\lambda \in \mathbb{C} \setminus \mathcal{W}$ are either eigenvalues of H or in $\mathcal{R}$, i. e., there is no essential (continuous or residual) spectrum in $\mathbb{C} \setminus \mathcal{W}$. We note that $\max \operatorname{Re} \mathcal{W} \ll -1$ for physically sensible parameter constellations. Let $\zeta \in \mathbb{L}^2([0, L]; \mathbb{C}^2)$. We denote the solution φ of the inhomogeneous boundary value problem

$$\begin{aligned} &\begin{bmatrix} -\partial_z + \beta + \chi(\lambda) - \lambda & -i\kappa \\ -i\kappa & \partial_z + \beta + \chi(\lambda) - \lambda \end{bmatrix} \varphi + \zeta = 0 \\ &\text{with b. c.} \quad \varphi_1(t, 0) = r_0 \varphi_2(t, 0), \quad \varphi_2(t, L) = r_L \varphi_1(t, L) \end{aligned} \tag{32}$$

by $R_1(\lambda)\zeta$. An expression for $R_1(\lambda)\zeta$ is

$$\begin{aligned} [R_1(\lambda)\zeta](z) = \; & \frac{1}{h(\lambda)} T(z, 0; \lambda) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} (r_L, -1) \int_0^L T(L, s; \lambda) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \zeta(s)\, ds - \\ & \int_0^z T(z, s; \lambda) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \zeta(s)\, ds. \end{aligned} \tag{33}$$

Hence, $R_1(\lambda) : \mathbb{L}^2([0, L]; \mathbb{C}^2) \to \mathbb{L}^2([0, L]; \mathbb{C}^2)$ is compact for $\lambda \in \mathcal{R}$. The resolvent of H, $R(\lambda) := (\lambda Id - H)^{-1} : X \to X$ for $\lambda \in \mathcal{R}$ is

$$R(\lambda) \begin{pmatrix} \psi \\ p \end{pmatrix} = \begin{pmatrix} R_1(\lambda) \left(\psi + \frac{\rho p}{\lambda - i\Omega_r + \Gamma} \right) \\ \frac{1}{\lambda - i\Omega_r + \Gamma} \left[p + \Gamma R_1(\lambda) \left(\psi + \frac{\rho p}{\lambda - i\Omega_r + \Gamma} \right) \right] \end{pmatrix} \tag{34}$$

which is a compact perturbation of the operator $(\psi, p) \to (0, (\lambda - i\Omega_r + \Gamma)^{-1} p)$.

The following lemma provides an approximate upper bound for the real parts of the eigenvalues.

Lemma 11 *Let $\lambda \in \mathbb{C} \setminus \mathcal{W}$ be in the point spectrum of H. Then, λ is geomet-*

rically simple, and its real part satisfies the estimate

$$\operatorname{Re}\lambda \leq \Lambda_u := \max_{k=1\ldots m}\left\{-\frac{\Gamma_k}{2}, \operatorname{Re}\beta_k + 2\rho_k\right\}.$$

PROOF. Let (ψ, p) be an eigenvector associated to λ. Then, ψ is a multiple of $T(z, 0; \lambda)\left(\begin{smallmatrix} r_0 \\ 1 \end{smallmatrix}\right)$, and $p = \Gamma\psi/(\lambda - i\Omega_r + \Gamma)$. Thus, λ is geometrically simple. Partial integration of the eigenvalue equation (29) and its complex conjugate equation yields:

$$2\operatorname{Re}\lambda \leq 2\max_{k=1\ldots m}\left(\operatorname{Re}\beta_k + \operatorname{Re}\chi_k(\lambda)\right). \tag{35}$$

For $\operatorname{Re}\lambda > -\Gamma_k/2$, we get $\operatorname{Re}\chi_k(\lambda) \leq |\chi_k(\lambda)| \leq 2\rho$. $\square$

It turns out that we have to treat the cases $r_0 r_L = 0$ and $r_0 r_L \neq 0$ differently for more detailed analysis of the spectrum of H and the growth properties of the semigroup $T(t)$ generated by H.

5.1 The differentiable case: $r_0 r_L = 0$

According to the notations in [30], [31] we denote:

Definition 12 *A C_0 semigroup $T(t)$ is called eventually differentiable if there exists a $t_0 \geq 0$ such that $t \to T(t)x$ is differentiable for all $x \in X$ and $t > t_0$. It is called eventually compact if there exists a $t_0 \geq 0$ such that $T(t)$ is a compact operator for all $t > t_0$.*

Theorem 13 *If $r_0 r_L = 0$ in (28), then the C_0 semigroup $T(t)$ generated by H is eventually differentiable.*

PROOF. Let M, ω be such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. The constants M and ω exist since H generates a C_0 semigroup. According to [30], it is sufficient to find constants $a > 0$, $b > 0$, and $C > 0$ such that

(1) $\mathcal{R} \supset \Sigma(a, b) := \{\lambda : \; b\operatorname{Re}\lambda + \log|\operatorname{Im}\lambda| \geq a\}$, and
(2) $\|R(\lambda)\| \leq C|\operatorname{Im}\lambda|$ for all $\lambda \in \Sigma(a, b)$ with $\operatorname{Re}\lambda \leq \omega$.

See Figure 2 for an illustration how $\Sigma(a, b)$ looks like qualitatively.

Firstly, we prove property 1. We know that $\mathbb{C}_\omega := \{\lambda : \operatorname{Re}\lambda > \omega\} \subset \mathcal{R}$ because of $\|T(t)\| \leq Me^{\omega t}$. Consider the following two sets

$$\begin{aligned}\mathcal{S}_1 &:= \{\lambda : \operatorname{Im}\lambda > 1\} \setminus \mathbb{C}_\omega \\ \mathcal{S}_2 &:= \{\lambda : \operatorname{Im}\lambda < -1\} \setminus \mathbb{C}_\omega.\end{aligned}$$

Within each of both sets, we can choose the branch of the square root for γ_k satisfying

$$\lim_{|\lambda|\to\infty} \gamma_k(\lambda) - \mu_k(\lambda) = \lim_{|\lambda|\to\infty} \gamma_k(\lambda) - \lambda = 0. \tag{36}$$

Consider the function

$$\begin{aligned} \tilde{h}(\lambda) = &\ h(\lambda) \exp\left(- \sum_{k=1}^{m} \gamma_k(\lambda) l_k \right) \\ = &\ (r_L, -1) \prod_{k=m}^{1} \left(T_k(l_k; \lambda) e^{-l_k \gamma_k(\lambda)} \right) \begin{pmatrix} r_0 \\ 1 \end{pmatrix} \end{aligned} \tag{37}$$

which is a multiple of the characteristic function $h(\lambda)$ of H. (36) implies that the factor matrices $\tilde{T}_k(\lambda) = e^{-l_k\gamma_k(\lambda)} T_k(l_k; \lambda)$ of $\tilde{h}$ have the form

$$\tilde{T}_k(\lambda) = \begin{pmatrix} e^{-2l_k\gamma_k(\lambda)} & 0 \\ 0 & 1 \end{pmatrix} + A_k(\lambda)$$

where all coefficients of A_k satisfy the inequality

$$|A_{k,ij}(\lambda)| \leq c_k |\lambda|^{-1} e^{-2l_k \operatorname{Re} \lambda} \tag{38}$$

for some $c_k > 0$ in $\mathcal{S}_1$ and in $\mathcal{S}_2$. Hence, we can expand the matrix product in (37) into a sum such that $\tilde{h}(\lambda)$ reads:

$$\tilde{h}(\lambda) = r_0 r_L \exp\left(\sum_{k=1}^{m} \gamma_k(\lambda) l_k \right) - 1 + r(\lambda).$$

The first summand is zero and the remainder $r(\lambda)$ is bounded by

$$|r(\lambda)| \leq c|\lambda|^{-1} e^{-2L \operatorname{Re} \lambda} \tag{39}$$

for some $c > 0$ in $\mathcal{S}_1$ and $\mathcal{S}_2$. If we choose $b > 2L$, then

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Sigma(a,b)}} |\lambda|^{-1} e^{-2L \operatorname{Re} \lambda} = 0 \qquad \text{for all } a > 0.$$

Thus, we can choose a sufficiently large such that $\Sigma(a, b) \setminus \mathbb{C}_\omega \subset \mathcal{S}_1 \cup \mathcal{S}_2$ and

$$c|\lambda|^{-1} e^{-2L \operatorname{Re} \lambda} < 1/2 \quad \text{for all } \lambda \in \Sigma(a, b) \setminus \mathbb{C}_\omega.$$

Hence, $|r(\lambda)| < 1/2$, and $|\tilde{h}(\lambda)| > 1/2$ for all $\lambda \in \Sigma(a, b) \setminus \mathbb{C}_\omega$. Consequently, $\Sigma(a, b) \subset \mathcal{R}$.

Concerning property 2: The only term which is unbounded w.r.t. λ for $|\lambda| \to \infty$ in the right-hand-side of (34) is $R_1(\lambda)$. We substitute $h(\lambda) = \tilde{h}(\lambda) \exp\left(\sum_{k=1}^{m} l_k \gamma_k(\lambda)\right)$ in (33) and estimate

$$|T_k(z; \lambda)| \leq c e^{-l_k \operatorname{Re} \lambda} \tag{40}$$

for all $\lambda \in \mathcal{S}_1$ and $\mathcal{S}_2$ due to (36). (40) and $\tilde{h}(\lambda) > 1/2$ imply

$$\|R_1(\lambda)\| \leq c e^{-3L \operatorname{Re} \lambda} \tag{41}$$

for all $\lambda \in \mathcal{S}_1$ and $\mathcal{S}_2$. Hence, if we choose $b > 3L$ in the definition of $\Sigma(a, b)$, property 2 is also satisfied in $\Sigma(a, b)$. $\square$

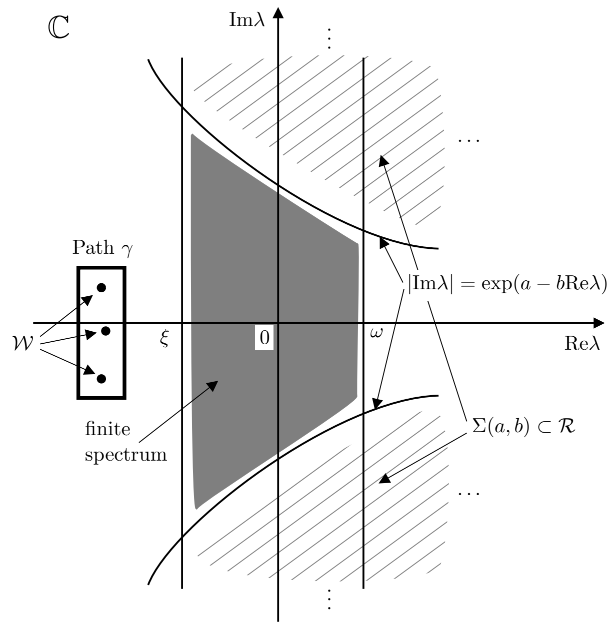

Fig. 2. Spectrum for the differentiable case. The sketch illustrates the location of the path γ and the set $\Sigma(a, b)$ in the complex plane constructed in the proof of Theorem 13.

The next theorem establishes precisely how the growth properties of the semigroup $T(t)$ are related to the spectrum of H.

Theorem 14 *Let $\xi > \max \operatorname{Re} \mathcal{W}$, and denote $\mathbb{C}_\xi := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \xi\}$, and $\sigma_+ := \operatorname{spec} H \cap \mathbb{C}_\xi$. Then, σ_+ consists of at most finitely many eigenvalues of H. All eigenvalues $\lambda \in \sigma_+$ have only finite algebraic multiplicity. The space X can be decomposed into two closed subspaces $X_1 \oplus X_2$ invariant with respect to H and $T(t)$ such that*

(1) $\dim X_1 < \infty$, $\operatorname{spec} H|_{X_1} = \sigma_+$ *and X_1 is spanned by the finitely many generalized eigenvectors of H associated to the eigenvalues of H in σ_+.*
(2) There exists a $M > 0$ such that $\|T(t)|_{X_2}\| \leq M e^{\xi t}$ for all $t > 0$.

PROOF. See also Figure 2 for an illustration of the spectral splitting. Let $\gamma \in \mathbb{C} \setminus \mathbb{C}_\xi$ be a smooth closed path around $\mathcal{W}$. Since the spectrum of H is discrete in $\mathbb{C} \setminus \mathcal{W}$, we can choose γ such that $\gamma \subset \mathcal{R}$. Define the projectors

$$\begin{aligned} P &:= \frac{1}{2\pi i} \oint_\gamma R(\lambda)\, d\lambda \\ Q &:= Id - P. \end{aligned}$$

These projectors decompose X into two closed subspaces $X_P = \operatorname{Im} P$, and $X_Q = \operatorname{Im} Q$ which are invariant with respect to H. The resolvent of $H|_{X_Q}$, $QR(\lambda)$, is compact since

$$Q \begin{pmatrix} 0 \\ (\lambda - i\Omega_r + \Gamma)^{-1} p \end{pmatrix} = 0,$$

and $R_1(\lambda)$ is compact. Since $T(t)$ is eventually differentiable, there exists a t_0 such that $T(t)$ is continuous with respect to t in the uniform operator topology for all $t \geq t_0$, i.e., $\|T(t+h) - T(t)\| \to_{h \to 0} 0$ for all $t \geq t_0$ [30]. Thus, $T(t)|_{X_Q}$ is continuous with respect to t in the uniform operator topology for all $t \geq t_0$. Consequently, $T(t)|_{X_Q}$ is eventually compact, i.e., compact for $t \geq t_0$ [30]. This permits us to split the closed subspace X_Q further: At most finitely many eigenvalues of $H|_{X_Q}$, the generator of $T(t)|_{X_Q}$, are situated in $\mathbb{C}_\xi$, and they have at most finite algebraic multiplicity [31]. We denote the corresponding finite-dimensional eigenspace by X_1, and its invariant closed complement by $X_{2,Q}$. Then, the spaces X_1 and $X_2 = X_P \oplus X_{2,Q}$ satisfy the assertions of the theorem: H_{X_P} is a bounded operator, and its spectrum outside the discrete set $\mathcal{W}$ is discrete. Hence, the growth of $T(t)|_{X_P}$ is restricted by $\|T(t)|_{X_P}\| \leq Me^{\xi t}$ for some $M > 1$ as the path γ is contained in $\mathbb{C} \setminus \mathbb{C}_\xi$. Likewise, the growth of the eventually compact semigroup $T(t)|_{X_{2,Q}}$ is bounded by the spectral bound of $H|_{X_{2,Q}}$ which is less than ξ: $\|T(t)|_{X_{2,Q}}\| \leq Me^{\xi t}$ for some $M > 1$ [31]. □

5.2 The hyperbolic case: $r_0 r_L \neq 0$

In order to prove a theorem similar to Theorem 14 for the case $r_0 r_L \neq 0$, we treat the operator H as a perturbation of the operator

$$H_0 = \begin{pmatrix} \begin{bmatrix} -\partial_z + \beta & 0 \\ 0 & \partial_z + \beta \end{bmatrix} & 0 \\ 0 & i\Omega_r - \Gamma \end{pmatrix}$$

defined in $Y \subset X$ (see also [28], [21], [22]). The spectrum of H_0 consists of $\mathcal{W}$ and the sequence of simple eigenvalues

$$\lambda_j^0 := \frac{1}{L} \left[\sum_{k=1}^{m} \beta_k l_k + \frac{1}{2} \log(r_0 r_L) + j\pi i \right] \text{for } j \in \mathbb{Z}.$$

The eigenvector of H_0 associated to λ_j^0 is

$$b_j^0 := \left(e^{\left(-\lambda_j^0 z + \int_0^z \beta(z)\,dz\right)} r_0, e^{\left(\lambda_j^0 z - \int_0^z \beta(z)\,dz\right)}, 0, 0 \right)^T .$$

The sequence $\{b_j^0 : j \in \mathbb{Z}\}$ establishes a basis of $\mathbb{L}^2([0, L]; \mathbb{C}^2) \times \{0\}$, i.e., there exists an automorphism of X mapping an orthonormal basis of $\mathbb{L}^2([0, L]; \mathbb{C}^2) \times \{0\}$ onto $\{b_j^0 : j \in \mathbb{Z}\}$. Firstly, we prove an estimate for the location of the

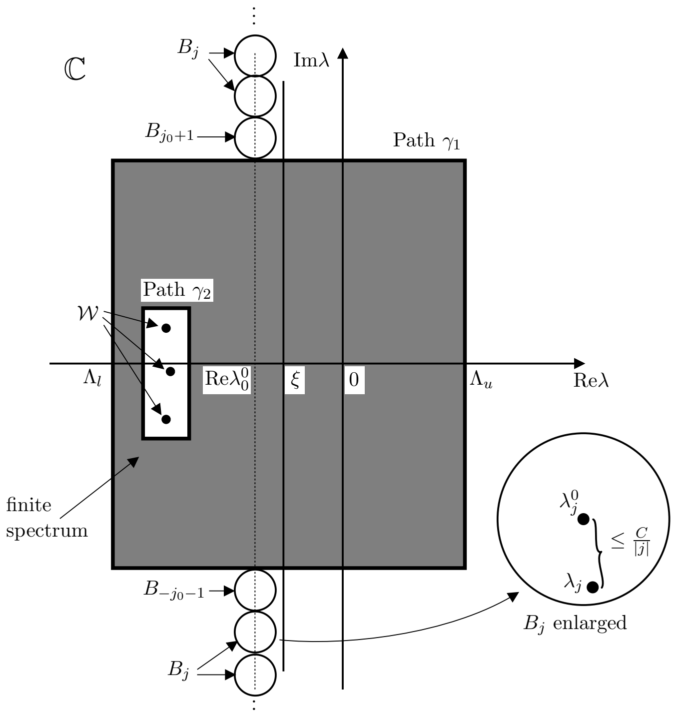

Fig. 3. Spectrum in the hyperbolic case. The sketch illustrates the location of the paths γ_1 and γ_2 constructed in the proof of Theorem 17 and the balls B_j around λ_j^0 containing the eigenvalues λ_j of H for large $|j|$ described in Lemma 15.

eigenvalues of H (see Lemma 11 for the definition of Λ_u and Figure 3 for illustration):

Lemma 15 *Let $r_0 r_l \neq 0$. Then, there exists a vertical strip $\mathcal{S} := \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \in [\Lambda_l, \Lambda_u]$ such that $\mathrm{spec}\, H \subset \mathcal{S}$. There exist constants $R > 0$ and $C > 0$ such that the following holds:*

(1) If λ is an eigenvalue of H and $|\lambda| > R$, then λ is algebraically simple and there exists a $j \in \mathbb{Z}$ such that $|\lambda - \lambda_j^0| < C/|j| < \pi/(2L)$.
(2) If $|\lambda_j^0| > R$, then there is exactly one eigenvalue of H in the ball B_j of radius $\pi/(2L)$ around λ_j^0.

PROOF. We choose the branch of the square root such that $\gamma_k(\lambda) - \mu_k(\lambda) \to 0$ and $\gamma_k(\lambda) - \lambda \to 0$ for $|\lambda| \to \infty$ in the negative half-plane of $\mathbb{C}$. Hence,

$e^{2l_k\gamma_k(\lambda)} \rightarrow_{\operatorname{Re}\lambda\to-\infty} 0$. Consequently, the matrices

$$e^{l_k\gamma_k(\lambda)}T_k(l_k;\lambda) \rightarrow_{\operatorname{Re}\lambda\to-\infty} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Accordingly, the multiple of the characteristic function of H converges for $\operatorname{Re}\lambda \to -\infty$:

$$\exp\left(\sum_{k=1}^{m} l_k\gamma_k(\lambda)\right) h(\lambda) \rightarrow_{\operatorname{Re}\lambda\to-\infty} r_0 r_L \neq 0,$$

and this limit is uniform for $\operatorname{Im}\lambda$. Consequently, there exists a $\Lambda_l < 0$ such that $h(\lambda) \neq 0$ if $\operatorname{Re}\lambda < \Lambda_l$. The upper limit for the strip $\mathcal{S}$ has been constructed in Lemma 11.

Consider the function

$$h_0(\lambda) = r_0 r_L \exp\left(\sum_{k=1}^{m} \beta_k l_k - \lambda L\right) - \exp\left(-\sum_{k=1}^{m} \beta_k l_k + \lambda L\right).$$

The characteristic function h converges to h_0 within the vertical strip $\mathcal{S}$ for $|\operatorname{Im}\lambda| \to \infty$:

$$|h(\lambda) - h_0(\lambda)| \leq C/|\operatorname{Im}\lambda| \qquad \text{for } \lambda \in \mathcal{S} \text{ and some } C > 0. \tag{42}$$

The function h_0 has the period 2π with respect to $\operatorname{Im}\lambda$, and its roots are λ_j^0 ($j \in \mathbb{Z}$). Outside of the neighborhood of the roots λ_j^0, $|h_0|$ is uniformly bounded from below within $\mathcal{S}$: $|h_0| > c > 0$. Furthermore,

$$h_0'(\lambda_j^0) = (-1)^{j+1} 2L\sqrt{r_0 r_L} \neq 0.$$

Hence, all λ_j^0 are uniformly simple roots of h_0. Since h and h_0 are analytic in $\mathcal{S} \setminus \mathcal{W}$, the convergence (42) implies the assertions 1 and 2 of the lemma. □

Corollary 16 *There exists a ball B, and constants $j_0 \geq 0$ and $C > 0$ such that there is a one-to-one correspondence between eigenvalues of H in $\mathbb{C} \setminus B$ and the elements of $\{\lambda_j^0 : |j| \geq j_0\}$. If we denote the eigenvalue corresponding to λ_j^0 by λ_j, then the eigenvector b_j associated to λ_j satisfies*

$$\left\| b_j - b_j^0 \right\| \leq \frac{C}{|j|}$$

if b_j is scaled appropriately.

PROOF. If we choose B around 0 of radius R according to Lemma 15, then we can associate the eigenvalue of H located in the ball $B_j = B_{\pi/(2L)}(\lambda_j^0)$ to λ_j^0.

The eigenvector b of H associated to λ can be scaled such that it has the form

$$b(z) = \begin{pmatrix} T(z,0;\lambda)\begin{pmatrix} r_0 \\ 1 \end{pmatrix} \\ \frac{\Gamma(z)}{\lambda - i\Omega_r(z)+\Gamma(z)} T(z,0;\lambda)\begin{pmatrix} r_0 \\ 1 \end{pmatrix} \end{pmatrix}. \tag{43}$$

Within the strip $\mathcal{S}$, the expressions $e^{\pm l_k \gamma_k(\lambda)}$ are uniformly bounded, and we can choose a branch of the square root such that $\gamma_k(\lambda) - \lambda \to_{\operatorname{Im}\lambda\to\infty} 0$, and $\gamma_k(\lambda) - \mu_k(\lambda) \to_{\operatorname{Im}\lambda\to\infty} 0$. Hence, the off-diagonal terms of each matrix T_k are of order $O(|\operatorname{Im}\lambda|^{-1})$, and the diagonal terms have the form $e^{\pm(\beta_k-\lambda)z} + O(|\operatorname{Im}\lambda|^{-1})$. $\square$

We can now state a theorem similar to Theorem 14:

Theorem 17 *Let $r_0 r_L \neq 0$, and $\xi > \max\{\max \operatorname{Re}\mathcal{W}, \operatorname{Re}\lambda_0^0\}$. Then, the space X can be decomposed into two closed subspaces $X_1 \oplus X_2$ which are invariant with respect to H and have the following properties:*

(1) $\dim X_1 < \infty$, and X_1 is spanned by at most finitely many generalized eigenvectors of H.
(2) There exists a $M > 0$ such that $\|T(t)|_{X_2}\| \leq Me^{\xi t}$ for all $t \geq 0$.

PROOF. We define the family of operators $Y \to X$

$$H_\theta = \begin{pmatrix} \begin{bmatrix} -\partial_z + \beta(n) & -i\theta\kappa \\ -i\theta\kappa & \partial_z + \beta(n) \end{bmatrix} & \theta\rho \\ \theta\Gamma & i\Omega_r - \Gamma \end{pmatrix}.$$

The operator H corresponds to $\theta = 1$ and H_0 to $\theta = 0$. The strip $\mathcal{S}$, the ball B and the constants j_0 and C from Lemma 15 and Corollary 16 can be chosen uniformly for the family of operators H_θ.

Since $\{b_j^0 : j \in \mathbb{Z}\}$ is a basis of $\mathbb{L}^2([0,L];\mathbb{C}^2) \times \{0\}$ [28], [22], there exists a constant c such that for any sequence $(x_j) \in \ell^2$ the inequality $c\sum_{j\in\mathbb{Z}} |x_j|^2 \leq \|\sum_{j\in\mathbb{Z}} x_j b_j^0\|^2$ holds.

We choose the constant j_0 sufficiently large such that Lemma 15 and Corollary 16 hold for j_0, $\operatorname{Re}\lambda_j < \xi$ for all $|j| > j_0$, and such that

$$\sum_{|j|>j_0} \|b_j - b_j^0\|^2 < c. \tag{44}$$

Next, we define the rectifiable path γ_1 as the border of the rectangle $[\Lambda_l + i(\operatorname{Im}\lambda^0_{j_0} + \pi/(2L)),\ \Lambda_l + i(\operatorname{Im}\lambda^0_{-j_0} - \pi/(2L)),\ \Lambda_u + i(\operatorname{Im}\lambda^0_{-j_0} - \pi/(2L)),\ \Lambda_u + i(\operatorname{Im}\lambda^0_{j_0} + \pi/(2L))]$. Thus, γ_1 is located in the resolvent set of H_θ for all $\theta \in [0,1]$. See also Figure 3 for an illustration. The spectral projections

$$P_\theta := \frac{1}{2\pi i}\oint_{\gamma_1} (\lambda Id - H_\theta)^{-1}\,d\lambda \qquad Q_\theta := Id - P_\theta$$

split X into the closed subspaces $X_{P,\theta} = \operatorname{Im} P_\theta$ and $X_{Q,\theta} = \operatorname{Im} Q_\theta$ which are invariant with respect to H_θ.

Next, we will construct a map $K : X \to X$ which is injective, a compact perturbation of Id in X and maps $X_{Q,0}$ into $X_{Q,1}$ by mapping $b^0_j \to b_j$ for $|j| > j_0$:

The projections P_θ and Q_θ depend continuously on θ. Define a sufficiently fine mesh $\{\theta_l : l = 0\ldots N\}$ such that $\|P_{\theta_l} - P_{\theta_{l-1}}\| < 1$ for all $l = 1\ldots N$. Then $P_l + Q_{l-1}$ and $P_{l-1} + Q_l$ are automorphisms of X. Moreover, they are compact perturbations of Id since the resolvent $(\lambda Id - H_\theta)^{-1}$ is a compact perturbation of the operator $(\psi, p) \to (0, (\lambda - i\Omega_r + \Gamma)^{-1}p)$. Let $J := \prod_{l=N}^{1}(P_{\theta_l} + Q_{\theta_{l-1}})$, and $\tilde{J} := \prod_{l=1}^{N}(Q_{\theta_l} + P_{\theta_{l-1}})$. J and $\tilde{J}$ are automorphisms of X, and compact perturbations of Id. J maps injectively $X_{P,0}$ into $X_{P,1}$, and $\tilde{J}$ maps injectively $X_{P,1}$ into $X_{P,0}$. Thus, J is an isomorphism from $X_{P,0}$ onto $X_{P,1}$. We define K in the following way: Let $x = \sum_{|j|>j_0} x_j b^0_j + x_P$ where $x_P \in X_{P,0}$. Then, $Kx := \sum_{|j|>j_0} x_j b_j + Jx_P$. K is injective due to (44) and since J is injective, and K is a compact perturbation of Id [35].

Consequently, K is also surjective. Hence, it maps $X_{Q,0}$ *onto* $X_{Q,1}$, i. e. the set $\{b_j : |j| > j_0\}$ establishes a $\mathbb{L}^2$ basis of $X_{Q,1}$. This implies that there exists a $M > 0$ such that $\|T(t)|_{X_{Q,1}}\| \le M^{\xi t}$ since $\operatorname{Re}\lambda_j < \xi$ for all $|j| > j_0$.

Let γ_2 be a smooth closed path in $\mathcal{R}$ encircling $\mathcal{W}$, and situated in the half-plane $\{\lambda : \operatorname{Re}\lambda < \xi\}$ and in the interior of γ_1. Define the spectral projection

$$P_2 := \frac{1}{2\pi i}\oint_{\gamma_2} R(\lambda)\,d\lambda,$$

and its image by $X_{\mathcal{W}}$. $H|_{X_{\mathcal{W}}}$ is a bounded operator which has a discrete spectrum outside of $\mathcal{W}$. Hence, there exists a $M > 0$ such that $\|T(t)|_{X_{\mathcal{W}}}\| \le Me^{\xi t}$. Moreover, the projections P_1 and P_2 commute, and the image of $P_1 - P_2$ is finite-dimensional since the spectrum of H is discrete between the paths γ_1 and γ_2.

Consequently, we can define $X_1 = \operatorname{Im}(P_1 - P_2)$, and $X_2 = X_{Q,1} \oplus X_{\mathcal{W}}$ to meet the assertions of the theorem. $\square$

The Theorems 14 and 17 assert basically the same growth properties for the semigroup $T(t)$ despite the different constructions. We collect both results in the following corollary.

Corollary 18 *Denote*

$$\xi_0 := \begin{cases} \max\{\operatorname{Re}\lambda_0^0, \max\operatorname{Re}\mathcal{W}\} & \text{if } r_0 r_L \neq 0, \\ \max\operatorname{Re}\mathcal{W} & \text{if } r_0 r_L = 0. \end{cases}$$

Let $\xi > \xi_0$. Then, there are at most finitely many eigenvalues of H of finite algebraic multiplicity in the right half-plane $\mathbb{C}_\xi := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \xi\}$. Moreover, X can be decomposed into two $T(t)$-invariant subspaces

$$X = X_+ \oplus X_-$$

where X_+ is at most finite-dimensional and spanned by the generalized eigenvectors associated to the eigenvalues of H in $\mathbb{C}_\xi$. There exists a constant M such that the restriction of $T(t)$ to X_- is bounded according to

$$\|T(t)|_{X_-}\| \leq Me^{\xi t} \tag{45}$$

in any norm which is equivalent to the X-norm.

Remark: The eigenvalues of H can be computed numerically by solving the complex equation $h(\lambda) = 0$. The eigenvalues of H_0 in $\mathbb{C}\setminus\mathcal{W}$ form the sequence λ_j^0 for $\kappa = 0$, $\rho = 0$, $r_0^0 r_L^0 \neq 0$ (see Theorem 17). The roots of the characteristic function h can be obtained by continuing along the parameter path $\theta\kappa$, $\theta\rho$, $r_0^0 + \theta(r_0 - r_0^0)$, $r_L^0 + \theta(r_L - r_L^0)$ for $\theta \in [0, 1]$.

6 Existence and properties of the finite-dimensional center manifold

In this section we construct a low-dimensional attracting invariant manifold for system (12), (14) using the general theorems about the persistence and properties of normally hyperbolic invariant manifolds in Banach spaces [17], [18], [19]. The statements of the theorem and the proofs rely only on the system's structure

$$\begin{aligned} \frac{d}{dt}E &= H(n)E \\ \frac{d}{dt}n_k &= \varepsilon(F_k(n_k) - g_k(n_k)[E, E]) \quad \text{for } k = 1, \ldots, m, \end{aligned} \tag{46}$$

the spectral properties of H for fixed n, the smoothness of the semiflow $S(t; \cdot)$ generated by (46) with respect to parameters and initial values, and the smallness of ε. In addition to the results of Corollary 18 we make the following assumption about the spectrum of H and its dependence on n:

Assumption 19 (Uniform spectral gap at imaginary axis) *Assume there exists a simple connected compact set $\mathcal{K} \subset \mathbb{R}^m$ with the following properties:*

(1) The constant ξ_0 defined in Corollary 18, which now depends on n is uniformly bounded from above by a constant less than zero, i.e., there exists a constant c independent of $n \in \mathcal{K}$ such that

$$\xi_0(n) \leq -c < 0 \quad \textit{for all } n \in \mathcal{K}. \tag{47}$$

(2) There exists a constant $\xi \in (-c, 0)$ independent of $n \in \mathcal{K}$ such that the spectrum of $H(n)$ can be split uniformly for all $n \in \mathcal{K}$ into a non-empty non-negative part and a part with real part less than ξ:

$$\begin{aligned} &\operatorname{spec} H(n) = \sigma_c(n) \cup \sigma_s(n) \qquad \textit{where} \\ &\quad \operatorname{Re}\sigma_c(n) \geq 0 \\ &\quad \operatorname{Re}\sigma_s(n) < \xi < 0. \end{aligned}$$

Assumption 19 asserts that there exists a set of n such that $H(n)$ has a uniform spectral gap at the imaginary axis. In general, this can only be verified by actually computing the eigenvalues of $H(n)$ and their dependence on n numerically. However, the following lemma illustrates that Assumption 19 is natural in the sense that it is a consequence of the existence of nontrivial dynamics that is bounded uniformly for $\varepsilon \to 0$. In Lemma 20, we consider system (46) as a family of evolution equations depending on the parameter ε in an interval $[0, \varepsilon_0)$.

Lemma 20 *Assume that there exist a one-parameter family of trajectories $(E(t;\varepsilon), n(t;\varepsilon))$ ($t \geq 0$) of system (46), and a compact set $N \subset (\underline{n}, \infty)^m$ and constants $E_u \geq E_l > 0$, and $c > 0$ that do not depend on $\varepsilon \in [0, \varepsilon_0)$ with the following properties:*

(1) $\xi_0(n(0,\varepsilon)) \leq -c < 0$,
(2) $n(0;\varepsilon) \in N$, and
(3) $\|E(t;\varepsilon)\| \in [E_l, E_u]$ for all $t \geq 0$.

Then, H satisfies Assumption 19.

PROOF. Since N is compact, there exists a sequence $\varepsilon_k \to_{k\to\infty} 0$ such that $n(0, \varepsilon_k)$ converges to some $n_0 \in N$. The value of ξ_0 depends continuously on n. Hence, $\xi_0(n_0) \leq -c$. If $\max \operatorname{Re} \operatorname{spec} H(n_0) \geq 0$, Corollary 18 implies Assumption 19 for $\mathcal{K} = \{n_0\}$. Thus, we have to show only that $\max \operatorname{Re} \operatorname{spec} H(n_0) < 0$ contradicts the assumptions about the bounds for $E(t;\varepsilon)$.

Let us assume $\max \operatorname{Re} \operatorname{spec} H(n_0) < 0$. We denote the semigroup generated by $H(n_0)$ by $T_0(t)$. Then, the estimate (45) in Corollary 18 is satisfied for the whole semigroup $T_0(t)$ with some $M \geq 1$ and $\xi < 0$:

$$\|T_0(t)\| \leq M e^{\xi t} \quad \text{for } t \geq 0.$$

We choose a time $t_e > \xi^{-1}(\log E_l - \log(M E_u))$. Thus, $\|T_0(t_e)\| E_u < E_l$. The semiflow $S(t; (E, n))$ generated by (46) depends continuously on ε in $\varepsilon = 0$ and on (E, n) (see Corollary 10 and the remarks thereafter). At $\varepsilon = 0$, $S(t; (E, n_0)) = (T_0(t)E, n_0)$ for all $E \in X$. Consequently, $\|E(t_e; \varepsilon_k) - T_0(t_e)E(0; \varepsilon_k)\| \to_{k \to \infty} 0$. Hence, $\|E(t_e; \varepsilon_k)\| < E_l$ if $\|E(0; \varepsilon_k)\| < E_u$ for sufficiently large k. This contradicts the uniform boundedness of $E(t; \varepsilon)$. $\square$

A practically relevant example for the type of uniformly bounded dynamics assumed to exist in Lemma 20 are relative equilibria, that is, solutions of the type $(E(t), n(t)) = (E_0 e^{i\omega t}, n_0)$. The location of relative equilibria does not depend on ε. Numerical evidence shows that there exist relative equilibria satisfying the first point of Assumption 19 for the set $\mathcal{K} = \{n_0\}$ for physically sensible parameters, that is, $\kappa_k \neq 0$ or $\rho_k \neq 0$ for at least one $k \in \{1, \ldots, m\}$. Since $i\omega \in \operatorname{spec} H(n_0)$ for a relative equilibrium $(E_0 e^{i\omega t}, n_0)$, the non-negative part $\sigma_c(n_0)$ of $\operatorname{spec} H(n_0)$ is non-empty. Indeed, $\sigma_c(n)$ is situated on the imaginary axis in all practically relevant cases [7,1,16].

Due to Corollary 18, the number of elements of $\sigma_c(n)$ is finite and, hence, constant in $\mathcal{K}$ if the eigenvalues are counted according to their algebraic multiplicity. We denote this number by q. Moreover, for each $\gamma \in [\xi, 0)$ there exists a bounded simple connected open set $U_\gamma \supset \mathcal{K}$ with rectifiable boundary such that the splitting of $\operatorname{spec} H(n)$ can be can be extended to U_γ in the following manner:

$$\begin{aligned} \operatorname{spec} H(n) = {} & \sigma_c(n) \cup \sigma_s(n) && \text{where} \\ \operatorname{Re} \sigma_c(n) > {} & \gamma, && \\ \operatorname{Re} \sigma_s(n) < {} & \xi && \text{for all } n \in \operatorname{cl} U_\gamma. \end{aligned}$$

There exist spectral projections of $H(n)$, $P_c(n)$ and $P_s(n) \in \mathcal{L}(X)$, corresponding to this splitting. They are well defined and unique for all $n \in U_\xi$ and depend smoothly on n. We define the corresponding closed invariant subspaces of X by $X_c(n) = \operatorname{Im} P_c(n) = \ker P_s(n)$ and $X_s(n) = \operatorname{Im} P_s(n) = \ker P_c(n)$. The complex dimension of $X_c(n)$ is q. Let $B(n) : \mathbb{C}^q \to X$ be a basis of $X_c(n)$ which depends smoothly on n. $B(\cdot)$ is well defined in U_ξ because U_ξ is simply connected, has rectifiable boundary and H has a uniform spectral splitting on $\operatorname{cl} U_\xi$. The existence of the basis B and the spectral projection P_c and their smooth dependence on $n \in U_\xi$ imply that the map $\mathcal{R} : X \times U_\xi \to \mathbb{C}^q \times U_\xi$ defined by

$$\mathcal{R}(E, n) := (B(n)^{-1} P_c(n) E, n)$$

is well defined and smooth. Using these notations, we can state the following theorem:

Theorem 21 (Model reduction)
Let $k > 2$ be an integer number, $\Delta \in (\xi, 0)$, and $\mathcal{N}$ be a closed bounded subset of $\mathbb{C}^q \times U_{\xi/k}$. Then, there exists an $\varepsilon_0 > 0$ such that the following holds. For all $\varepsilon \in [0, \varepsilon_0)$, there exists a C^k manifold $\mathcal{C} \subset X \times \mathbb{R}^m$ satisfying:

(i) (Invariance) *$\mathcal{C}$ is $S(t, \cdot)$-invariant relative to $\mathcal{R}^{-1}\mathcal{N}$. That is, if $(E, n) \in \mathcal{C}$, $t \geq 0$, and $S([0, t]; (E, n)) \subset \mathcal{R}^{-1}\mathcal{N}$, then $S([0, t]; (E, n)) \subset \mathcal{C}$.*

(ii) (Representation) *$\mathcal{C}$ can be represented as the graph of a map which maps*

$$(E_c, n, \varepsilon) \in \mathcal{N} \times [0, \varepsilon_0) \to ([B(n) + \varepsilon\nu(E_c, n, \varepsilon)]E_c, n) \in X \times \mathbb{R}^m$$

where $\nu : \mathcal{N} \times [0, \varepsilon_0) \to \mathcal{L}(\mathbb{C}^q; X)$ is C^{k-2} with respect to all arguments. Denote the X-component of $\mathcal{C}$ by

$$E_X(E_c, n, \varepsilon) = [B(n) + \varepsilon\nu(E_c, n, \varepsilon)]E_c \in X.$$

(iii) (Exponential attraction) *Let $\Upsilon \subset X \times \mathbb{R}^m$ be a bounded set with $\mathcal{R}\Upsilon \subset \mathcal{N}$ and a positive distance to the boundary of $\mathcal{N}$. Then, there exist a constant M and a time $t_c \geq 0$ with the following property: For any $(E, n) \in \Upsilon$ there exists a $(E_c, n_c) \in \mathcal{N}$ such that*

$$\|S(t + t_c; (E, n)) - S(t; (E_X(E_c, n_c, \varepsilon), n_c))\| \leq Me^{\Delta t} \tag{48}$$

for all $t \geq 0$ with $S([0, t + t_c]; (E, n)) \subset \Upsilon$.

(iv) (Flow) *The values $\nu(E_c, n, \varepsilon)E_c$ are in Y and their $P_c(n)$-component is 0 for all $(E_c, n, \varepsilon) \in \mathcal{N} \times [0, \varepsilon_0)$. The flow on $\mathcal{C} \cap \mathcal{R}^{-1}\mathcal{N}$ is differentiable with respect to t and governed by the following system of ODEs:*

$$\begin{aligned} \frac{d}{dt}E_c &= \left[H_c(n) + \varepsilon a_1(E_c, n, \varepsilon) + \varepsilon^2 a_2(E_c, n, \varepsilon)\nu(E_c, n, \varepsilon)\right] E_c \\ \frac{d}{dt}n &= \varepsilon F(E_c, n, \varepsilon) \end{aligned} \tag{49}$$

where

$$\begin{aligned} H_c(n) &= B(n)^{-1}H(n)P_c(n)B(n) \\ a_1(E_c, n, \varepsilon) &= -B(n)^{-1}P_c(n)\partial_n B(n)F(E_c, n, \varepsilon) \\ a_2(E_c, n, \varepsilon) &= B(n)^{-1}\partial_n P_c(n)F(E_c, n, \varepsilon)(Id - P_c(n)) \\ F(E_c, n, \varepsilon) &= (F_k(n_k) - g_k(n_k)[E_X(E_c, n_c, \varepsilon), E_X(E_c, n_c, \varepsilon)])_{k=1}^m . \end{aligned}$$

System (49) is symmetric with respect to rotation $E_c \to E_c e^{i\varphi}$ and ν satisfies the relation $\nu(e^{i\varphi}E_c, n, \varepsilon) = \nu(E_c, n, \varepsilon)$ for all $\varphi \in [0, 2\pi)$.

Remark: This theorem follows from the general results of [17], [18], [19]. In this case, the invariant manifold is even finite-dimensional and exponentially stable. The proof is mostly concerned with the proper definition of the coordinates and describes in detail the appropriate cut-off modification of the system outside of the region of interest to make the unperturbed invariant manifold compact. A similar result about model reduction for systems of ODEs with the structure (1) has been presented already by [32] using Fenichel's Theorem [33].

PROOF.
Existence, representation, and smoothness
Firstly, we introduce a splitting of $E \in X$ which is valid for $n \in U_\xi$. Let $n \in U_\xi$. For any $E \in X$, we define $E_c = B(n)^{-1}P_c(n)E \in \mathbb{C}^q$ and $E_s = P_s(n)E \in X_s(n)$. Then, $E = B(n)E_c + E_s$, and a decomposition of (12) by $B(n)^{-1}P_c(n)$ and $P_s(n)$ implies that $E_c \in \mathbb{C}^q$, $E_s \in X_s(n) \subset X$, and $n \in \mathbb{R}^m$ satisfy the system

$$\frac{d}{dt}E_c = H_c(n)E_c + a_{11}(E_c, E_s, n)E_c + a_{12}(E_c, E_s, n)E_s \tag{50}$$
$$\frac{d}{dt}E_s = H_s(n)E_s + a_{21}(E_c, E_s, n)E_c + a_{22}(E_c, E_s, n)E_s \tag{51}$$
$$\frac{d}{dt}n_k = f_k(E_c, E_s, n) \qquad \text{for } k = 1 \dots m \tag{52}$$

where $H_c, a_{11} : \mathbb{C}^q \to \mathbb{C}^q$, $a_{12} : X \to \mathbb{C}^q$, $a_{21} : \mathbb{C}^q \to X$, $a_{22} : X \to X$, and $H_s : Y \to X$ are linear operators defined by

$$\begin{aligned} H_c(n) &= B^{-1}HP_cB & H_s(n) &= HP_s - 2\xi P_c \\ a_{11}(E_c, E_s, n) &= -B^{-1}P_c\partial_n Bf & a_{12}(E_c, E_s, n) &= B^{-1}\partial_n P_c f P_s \\ a_{21}(E_c, E_s, n) &= -P_s\partial_n Bf & a_{22}(E_c, E_s, n) &= -P_c\partial_n P_c f P_s \end{aligned}$$
$$f_k(E_c, E_s, n) = \varepsilon\left(F_k(n_k) - g_k(n_k)[B(n)E_c + E_s, B(n)E_c + E_s]\right)$$

for $k = 1 \dots m$. We introduced the term $-2\xi P_c E_s$ which is 0 artificially in (51). System (50)–(52) couples a system of ODEs in $\mathbb{C}^q$, an evolution equation in X, and a system of ODEs in $\mathbb{R}^m$. The right-hand-side of (50)–(52) is only properly defined as long as n stays in U_ξ.

In the next step, we modify system (50)–(52) such that it is globally defined and generates a semiflow. Beforehand, we introduce some notation.

Let $d : \mathbb{R} \to [0, 1]$ be a smooth monotone function such that

$$d(x) = \begin{cases} 0 & x \le 0 \\ 1 & x \ge 1. \end{cases}$$

There exists a smooth and globally Lipschitz continuous map $N : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$N(n) = \begin{cases} n & \text{if there exists a } E_c \in \mathbb{C}^q \text{ such that } (E_c, n) \in \mathcal{N} \\ \in U_{\xi/k} & \text{otherwise.} \end{cases}$$

Using the map N we can modify the map $\mathcal{R}$ outside of $\mathcal{N}$, thus, extending it smoothly to the whole space $X \times \mathbb{R}^m$:

$$\tilde{\mathcal{R}}(E, n) := \mathcal{R}(E, N(n)).$$

Since $\tilde{\mathcal{R}}$ is identical to $\mathcal{R}$ on the set $\mathcal{N}$, $\tilde{\mathcal{R}}^{-1}\mathcal{N} \supseteq \mathcal{R}^{-1}\mathcal{N}$. Let $\sigma > 0$ and

$$\begin{aligned}
n_{\max} &:= \max_{(E_c,n)\in\mathcal{N}} |n| \\
E_{\max} &:= \max_{(E_c,n)\in\mathcal{N}} |E_c| \\
R &:= \sqrt{6 + E_{\max}^2 + n_{\max}^2}, \\
s(x, E_c, n) &:= |E_c|^2 + |n|^2 + x^2 - R^2 \quad \text{for } x \in \mathbb{R}, E_c \in \mathbb{C}^q, n \in \mathbb{R}^m, \\
D(E_c, n) &:= d\left(|E_c|^2 + |n|^2 - E_{\max}^2 - n_{\max}^2\right).
\end{aligned}$$

The functions s and D are smooth with respect to their arguments. Consider the following modification of system (50)–(52):

$$\begin{aligned}
\frac{d}{dt}E_c &= H_c(N(n))E_c + \tilde{a}_{11}E_c + \tilde{a}_{12}E_s \\
&\quad - D(E_c, n)\left[H_c(N(n))E_c + \tilde{a}_{11}E_c + \tilde{a}_{12}E_s + \sigma s(x, E_c, n)E_c\right] && (53) \\
\frac{d}{dt}E_s &= H_s(N(n))E_s + \tilde{a}_{21}E_c + \tilde{a}_{22}E_s && (54) \\
\frac{d}{dt}n_k &= \tilde{f}_k(E_c, E_s, n) - D(E_c, n)\left[\tilde{f}_k(E_c, E_s, n) + \sigma s(x, E_c, n)n_k\right] && (55)
\end{aligned}$$

for $k = 1 \dots m$, augmented by a differential equation for the auxiliary real variable x:

$$\frac{d}{dt}x = \tilde{g}(x, E_c) - \sigma s(x, E_c, n)x \qquad (56)$$

where

$$
\begin{aligned}
\tilde{a}_{11}(E_c,E_s,n) &= -B(N(n))^{-1}P_c(N(n))\partial_n B(N(n))\partial_n N(n)\tilde{f}(E_c,E_s,n)\\
\tilde{a}_{12}(E_c,E_s,n) &= B(N(n))^{-1}\partial_n P_c(N(n))\partial_n N(n)\tilde{f}(E_c,E_s,n)P_s(N(n))\\
\tilde{a}_{21}(E_c,E_s,n) &= -P_s(N(n))\partial_n B(N(n))\partial_n N(n)\tilde{f}(E_c,E_s,n)\\
\tilde{a}_{22}(E_c,E_s,n) &= -P_c(N(n))\partial_n P_c(N(n))\partial_n N(n)\tilde{f}(E_c,E_s,n)P_s(N(n))\\
\tilde{f}_k(E_c,E_s,n) &= \ f_k(E_c,E_s,N(n)) \text{ for } k=1\dots m,\\
\tilde{g}(x,E_c) &= \begin{cases}\left[-\frac{1}{2x}\frac{d}{dt}\left(|E_c|^2+|n|^2\right)\right]d(|x|-1) & \text{for } |x|>1\\ 0 & \text{for } |x|\le 1.\end{cases}
\end{aligned}
$$

The right-hand-side of system (53)–(56) is smooth and globally defined. It generates a semiflow $\tilde{S}(t;(E_c,E_s,n,x))$ on $\mathbb{C}^q\times X\times\mathbb{R}^m\times\mathbb{R}$. The modification has no effect if $(E_c,n)\in\mathcal{N}$. The equation for $\dot{x}$ implies

$$
\dot{s}=\begin{cases}-2\sigma s x^2 & \text{for } |x|\ge 2\\ -2\sigma s\left[(1-d(|x|-1))(|E_c|^2+|n|^2)+x^2\right] & \text{for } |x|<2\end{cases}
$$

in the vicinity of $\mathcal{M}_0:=\{(E_c,E_s,n,x):s(x,E_c,n)=0\}$. Thus $\mathcal{M}_0$ is an invariant manifold of $\tilde{S}$ which has an exponential attraction rate greater than 2σ. Moreover, system (53)–(56) implies:

$$
\frac{d}{dt}(P_c(N(n))E_s)=(\partial_n P_c\partial_n N\tilde{f}-2\xi Id)(P_c(N(n))E_s).
$$

Hence, the manifold $\mathcal{M}_1:=\{(E_c,E_s,n,x):P_c(N(n))E_s=0\}$ is invariant with respect to (53)–(56). For bounded E_c and E_s, the rate of attraction towards $\mathcal{M}_1$ is close to $2|\xi|$.

There is a one-to-one correspondence between the semiflows $S(t;\cdot)$ and $\tilde{S}(t,\cdot)$ in the following sense: The map acting from

$$
\begin{aligned}
&\{(E_c,E_s,n,x)\in\mathcal{M}_0\cap\mathcal{M}_1:(E_c,n)\in\mathcal{N}\}\to\ X\times U_{\xi/k} \text{ defined by}\\
&(E_c,E_s,n,x)\to\ (B(n)E_c+E_s,n)
\end{aligned}
\tag{57}
$$

is injective, Lipschitz continuous and maps $\tilde{S}$ onto S. The inverse

$$
(E,n)\to\left(B(n)^{-1}P_c(n)E,P_s(n)E,n,\sqrt{R^2-|B(n)^{-1}P_c(n)E|^2-|n|^2}\right)\tag{58}
$$

is properly defined in $\tilde{\mathcal{R}}^{-1}\mathcal{N}$.

At $\varepsilon=0$, $\tilde{f}$ and all $\tilde{a}_{ij}$ vanish. Hence,

$$
\tilde{\mathcal{C}}:=\{(E_c,E_s,n,x)\in\mathbb{C}^q\times X\times\mathbb{R}^m:E_s=0,s(x,E_c,n)=0\}
$$

is a smooth compact invariant manifold of (53)–(56). E_s decays with a rate greater than $|\xi|$. Hence, if $2\sigma>|\xi|$, the attraction rate transversal to $\tilde{\mathcal{C}}$ is greater than $|\xi|$. The generalized Lyapunov numbers for the component of the linearization of $\tilde{S}$ tangent to $\mathcal{C}$ are greater or equal than ξ/k. The perturbation to nonzero ε is C^1 small, and all derivatives of the perturbation with respect to (E_c, E_s, n, x), and ε up to order k are bounded uniformly for small ε in the vicinity of $\tilde{\mathcal{C}}$. Consequently, the general theorems of [17], [18], [19] imply:

There exists an ε_0 such that for all $\varepsilon \in [0, \varepsilon_0)$ there exists a compact invariant C^k manifold $\tilde{\mathcal{C}}$ for $\tilde{S}(t, \cdot)$. $\tilde{\mathcal{C}}$ is a C^1 small perturbation of $\tilde{\mathcal{C}}$. Hence, its E_s-component can be represented as a C^k graph

$$E_s = \eta_0(E_c, n, x, \varepsilon).$$

The contraction rates towards $\mathcal{M}_0$ and $\mathcal{M}_1$ are greater than $|\xi|$ close to $\tilde{\mathcal{C}}$. Consequently, $\tilde{\mathcal{C}} \subset \mathcal{M}_0 \cap \mathcal{M}_1$. The evolution of E_c, E_s and n does not depend on x if $(E_c, n) \in \mathcal{N}$. Hence, $\eta_0(E_c, n, x, \varepsilon)$ does not depend on x if $(E_c, n) \in \mathcal{N}$.

The existence of $\tilde{\mathcal{C}}$ and the one-to-one correspondence between S and $\tilde{S}$ imply that the manifold

$$\mathcal{C} := \{(B(n)E_c + \eta_0(E_c, n, \varepsilon), n) : (E_c, n) \in \mathcal{N}\}$$

is an invariant C^k manifold of S relative to $\tilde{\mathcal{R}}^{-1}\mathcal{N}$. The flow on $\mathcal{C}$ is governed by

$$\begin{aligned} \frac{d}{dt}E_c = &\ [H_c(n) + a_{11}(E_c, \eta_0(E_c, n, \varepsilon), n, \varepsilon)]\, E_c \\ &+ a_{21}(E_c, \eta_0(E_c, n, \varepsilon), n, \varepsilon)\eta_0(E_c, n, \varepsilon) \\ \frac{d}{dt}n_k = &\ f_k(E_c, \eta_0(E_c, n, \varepsilon), n). \end{aligned} \tag{59}$$

The rotational symmetry of the semiflow S implies

$$\eta_0(e^{i\varphi}E_c, n, \varepsilon) = e^{i\varphi}\eta_0(E_c, n, \varepsilon) \tag{60}$$

for all $(E_c, n, \varepsilon) \in \mathcal{N} \times [0, \varepsilon)$ and $\varphi \in [0, 2\pi)$.

Expansion of the graph η_0
The graph η_0 satisfies

$$\eta_0(E_c, n, 0) = 0 \qquad \text{for all } (E_c, n) \in \mathcal{N}. \tag{61}$$

Furthermore, the manifold $\mathcal{E} := \{(E, n) \in X \times U_{\xi/k} : E = 0\}$ is invariant with respect to S for positive ε. On $\mathcal{E}$, $\dot{E} = 0$, and $\dot{n}_k = \varepsilon F_k(n_k)$ for $k = 1 \ldots m$. Consequently, $\mathcal{E} \cap \tilde{\mathcal{R}}^{-1}\mathcal{N} \subset \mathcal{C}$, i.e.,

$$\eta_0(0, n, \varepsilon) = 0 \qquad \text{for } (0, n) \in \mathcal{N},\ \varepsilon \in [0, \varepsilon_0). \tag{62}$$

Finally, we observe that the right-hand-side of (53)–(56) depends smoothly on E_c and ε. Exploiting the identities (61) and (62), we may expand

$$\eta_0(E_c,n,\varepsilon) = \int_0^1 \partial_1\eta_0(sE_c,n,\varepsilon)\,ds E_c$$
$$= \varepsilon \int_0^1 \int_0^1 \partial_1\partial_3\eta_0(sE_c,n,r\varepsilon)\,dr\,dsE_c. \tag{63}$$

Denoting the double integral term in (63) by ν, we obtain

$$\eta_0(E_c,n,\varepsilon) = \varepsilon\nu(E_c,n,\varepsilon)E_c. \tag{64}$$

We obtain the assertion iv of the theorem by inserting (64) into system (59) for the flow on $\mathcal{C}$. The invariance of ν with respect to rotation of E_c is a direct consequence of (60).

Exponential attraction of $\mathcal{C}$
The theorems of [17], [18], [19] imply that the set of all points that stay in a small tubular neighborhood of a compact normally hyperbolic invariant manifold $\mathcal{M}$ for all $t \geq 0$ form a center-stable manifold which is foliated by stable fibers of attraction rate close to the generalized Lyapunov numbers in the stable part of the linearization of the semiflow along $\mathcal{M}$.

The existence of the map (58) on Υ and the evolution equation (54) for E_s imply that there exist constants C_1, C_2,and $\gamma > 0$ such that the inequality

$$\|P_s(n(s))E(s)\| \leq C_1e^{-\gamma s} + \varepsilon C_2 \int_0^s e^{-\gamma r}\,dr = C_1e^{-\gamma s} + \varepsilon\frac{C_2}{\gamma} \tag{65}$$

holds for all trajectories $S([0,t];(E,n)) \subset \Upsilon$ [30]. Consequently, there exists a time t_0 such that the map (58) maps $S(t_0;\Upsilon) \cap \mathcal{R}^{-1}\mathcal{N}$ into the small tubular neighborhood U of $\tilde{\mathcal{C}}$ that is foliated by stable fibers. This foliation implies that there exists a constant M_0 such that for all $u \in U$ there exists a fiber base point $u^* \in \tilde{\mathcal{C}}$ such that

$$\|\tilde{S}(t;u) - \tilde{S}(t;u_*)\| \leq M_0e^{\Delta t}. \tag{66}$$

We may have to decrease ε_0 (if necessary) in order to keep the decay rate at $|\Delta|$ in (66).

Let $t_1 \geq 0$ be such that $Me^{\Delta t_1}$ is less than the distance between the set $\mathcal{R}\Upsilon$ and the boundary of $\mathcal{N}$. Then, we can choose $t_c = t_0 + t_1$ to obtain assertion iii of the theorem: Let $(E,n) \in \Upsilon$ and $t \geq 0$ be such that $S([0,t+t_c];(E,n)) \subset \Upsilon$, and u be the image of $S(t_c;(E,n))$ under map (58). Then, $u \in U$, and, furthermore, the fiber base point $u_* = (E_c,E_s,n_c,x_c)$ for u satisfies $(E_c,n_c) \in \mathcal{N}$. Hence, inequality (66) implies the inequality (48) for (E_c,n_c) if we choose the constant M as M_0 multiplied by the Lipschitz constant of the map (57). $\square$

7 Practical application and possible generalizations of the model reduction theorem

Mode approximation The graph of the invariant manifold enters the description (49) of the flow on $\mathcal{C}$ only in the form $O(\varepsilon^2)\nu$. All other terms appearing in (49) can be expressed analytically as functions of the eigenvalues of $H(n)$. Systems of the form (49) but replacing ν by 0 are called *Mode approximation models*. These models are implicit systems of ODEs because the eigenvalues of H are given only implicitly as roots of the characteristic function h of H. The consideration of mode approximations has proven to be extremely useful for numerical and analytical investigations of longitudinal effects in multi-section semiconductor lasers because the dimension of system (49) is typically low (q is often either 1 or 2); see, e.g., [24], [36], [7], [1], [37], [38], [12], [6]. For illustration, Fig. 4 shows a two-parameter bifurcation diagram for a two-section laser that imitates an optical feedback experiment [6]: a laser (section S_1) is subject to optical feedback from the facet r_L of the passive section S_2. In the parameter range covered by the diagram the dimension of the invariant manifold $\mathcal{C}$ is 4 or less, ($m = 1$ since section S_2 is passive, $q = 2$). A detailed numerical comparison of Fig. 4 with simulation results for the PDE model (2)–(4) and more accurate models can be found in [14].

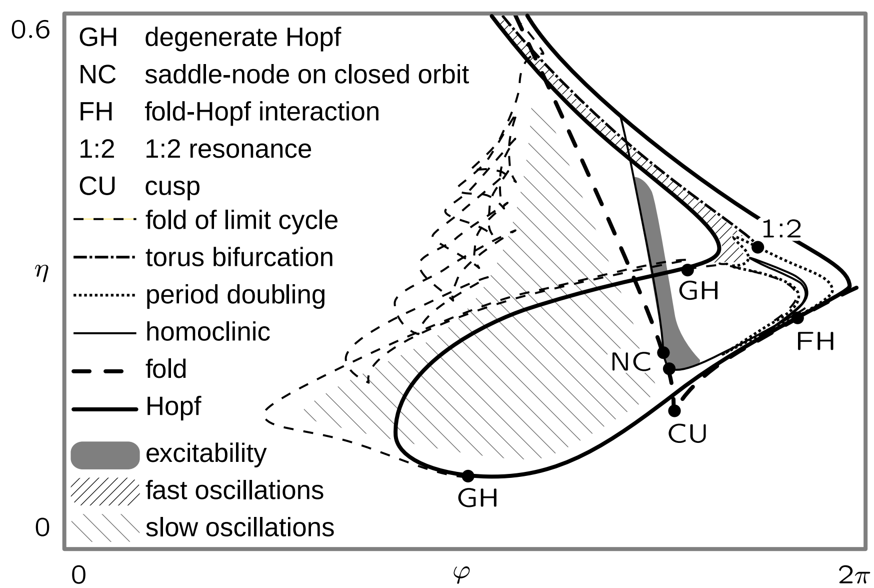

Fig. 4. Bifurcation diagram for the two-section laser investigated in [6]. The parameters are: $l_2 = 1.136$, $r_0 = 10^{-5}$, $r_L = \eta e^{i\varphi}$, $d_1 = -0.275$, $\kappa_1 = 3.96$, $\tilde{g}_1 = 2.145$ (linear gain model), $\alpha_1 = 5$, $\rho_1 = 0.44$, $\Gamma_1 = 90$, $\Omega_{r,1} = -20$, $I_1 = 6.757 \cdot 10^{-3}$, $\tau_1 = 359$, $\kappa_2 = \beta_2 = \rho_2 = 0$. The bifurcation parameters are the strength η and the phase φ of the feedback from the facet r_L of section S_2. In the experiment these parameters can be varied by changing the current in S_2. The highlighted dynamical regimes are of particular practical interest.

The Lang-Kobayashi system There is an obvious generalization of Theorem 21 to another class of laser models. A very popular model for the investigation of delayed optical feedback effects in semiconductor lasers is the Lang-Kobayashi system [39]; see, e.g., [23] and references therein. It reads

$$\begin{aligned} \frac{d}{dt}E(t) &= (1+i\alpha)nE(t) + \eta e^{i\varphi}E(t-1) \\ \frac{d}{dt}n(t) &= \varepsilon\left(F(n) - g(n)|E(t)|^2\right) \end{aligned} \tag{67}$$

if its scaling is appropriate to the situation of a short external cavity [40]. System (67) generates a semiflow in the Banach space $C([-1,0];\mathbb{C}) \times \mathbb{R}$ and has also the structure (1). The parameters have the same sense as in (2)–(4) (we have dropped the indices since there is only one section). The parameter ε is small if the external cavity is short. The operator H is a delay operator in (67). According to [31], Corollary 18 is also valid for the delay operator H (ξ_0 is $-\infty$ in Corollary 18). Moreover, the cut-off modification performed in the proof of Theorem 21 manipulates only the finite-dimensional components E_c and n. Hence, the proof does not rely on the ability to cut-off a smooth map smoothly in the infinite-dimensional space X which is the Hilbert space $X = \mathbb{L}^2([0,L];\mathbb{C}^2) \times \mathbb{L}^2([0,L];\mathbb{C}^2)$ in Section 6 but a Banach space for system (67). The only property of the operator $H(n)$ used in the proof is the existence of a spectral splitting according to Assumption 19 accompanied by the results of Corollary 18, and the smooth dependence of the dominating subspace X_c on n. Consequently, if Assumption 19 is satisfied, Theorem 21 applies to (67) as well. The set $\mathcal{K}$ supposed to exist in Assumption 19 is a point n_0 in $\mathbb{R}$ (typically referred to as *threshold carrier density*) in the case of a scalar n. Its existence can be shown analytically for the Lang-Kobayashi model (67).

There are other models in the spirit of (67) for different experimental situations, e.g., for lasers subject to dispersive feedback or for two lasers interacting with each other. All have the structure of (1) where H is a delay operator smoothly depending on n, and ε is small if the external cavity is short. Hence, Theorem 21 allows to reduce these models locally to low-dimensional systems of ODEs.

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